

Intermittent Superlinearity and the Finite-Time Singularity Problem

ABSTRACT

Self-reinforcing feedback appears across disciplines, and where it is superlinear it produces not unbounded growth but a finite-time singularity. The classical treatment assumes a fixed feedback exponent: for $dy/dx = k(1 + y^m)$, a singularity occurs if and only if $m > 1$, at the distance $x^* = (1/k)(\pi/m)/\sin(\pi/m)$. **This paper asks what happens when that exponent is not fixed but fluctuates across the critical value.** The question arose from an attempt to determine the exponent of a specific empirical curve, Bitcoin's historical price lows, where a flattening power law, a superlinear trajectory, and a noise-driven one proved difficult to distinguish over the observed range.

The main result is that the mean exponent does not decide the outcome. If the exponent process has a non-vanishing probability of sustained excursions above the critical value, together with a recurrence condition, the singularity occurs almost surely **whatever the mean may be**, with the time to the event bounded in expectation and its tail geometric. **No lower bound on the mean is required beyond its being positive:** a sufficiently high excursion, however brief and however rare, suffices, so the mean can lie arbitrarily close to zero. The proof rests on two elementary facts: the state is non-decreasing, so the level reached is never given back, and the duration of supercritical feedback still required to complete a singularity shrinks as that level rises. **A complementary result sharpens where the fluctuation must act:** a fluctuation entering the rate rather than the exponent cannot produce a singularity at any amplitude, however persistent. If the exponent process has a non-vanishing probability of sustained excursions above the critical value, together with a renewal condition, the singularity occurs almost surely **whatever the mean may be, including means below the critical value**, with the time to the event bounded in expectation and its tail geometric. The proof rests on two elementary facts: the state is non-decreasing, so the level reached is never given back, and the duration of supercritical feedback still required to complete a singularity shrinks as that level rises. Numerical results are given for mean exponents down to 0.80. **A complementary result sharpens where the fluctuation must act:** a fluctuation entering the rate rather than the exponent cannot produce a singularity at any amplitude, however persistent.

Three consequences bear on what any measurement of such a system can show, and none is favourable. Deep subcritical stretches flatten the early record toward a power law, so a system whose mean lies below the critical value can present a trajectory in which no feedback is detectable at all, and terminate regardless. **A singularity is completed only during a supercritical excursion**, so an exponent fitted to a completed event is drawn from above the critical value whatever the distribution behind it, and every level of the simulations here returns a supercritical value though each carries a subcritical base. **And in the terminal stretch the form is not diagnostic either:** a growing prefactor and a damping term that has been outgrown yield the same leading behaviour, so the approach to an event cannot distinguish this construction from a metastable escape over a threshold. What separates them lies in the quiet phase, where one decays and the other does not.

The second half of the paper concerns identification rather than dynamics. Explicit, checkable membership conditions are stated, including the requirement that the fluctuation reach the exponent and that an irreversible state variable be identified. Simulations of nested instances of the same equation, with code included, illustrate a difficulty that applies to every candidate system: **an observed quantity may follow none of the levels present**, so that an exponent fitted to a record belongs to no part of the system producing it. Two physical domains are examined against these conditions. In tokamak edge instabilities the currently established equation reduces to the same form but with a fixed quadratic exponent and a linear damping, placing them in a distinct class of metastable escape rather than in the one described here; the status of that fixed exponent, which is derived rather than measured, is discussed. In accelerating creep before slope failure, the governing relation used in that field coincides with the equation above, and its exponent is measured above the critical value rather than assumed. **Such a structure also resembles a fractal without being one**, since levels running the same equation at different rates have the same shape by construction, and the resemblance is supplied by the choice of observation window rather than found in the system.

A circuit realizing the equation is described, though not yet built. An oscillator runs between a fixed threshold and a rising one, so that its frequency is inversely proportional to the gap between them, and a second capacitor closes that gap as it accumulates. **The exponent such an arrangement produces is not a design choice.** It follows from how the gap closes and how the observable reads it, through a relation derived here that accounts for every case considered: charging once per cycle gives one value, charging at a constant rate gives another, and the same relation reproduces the exponents of two mechanical arrangements, one of which coincides with the isothermal and adiabatic compressions of elementary kinetic theory. **Every passive arrangement examined proves to be supercritical**, and the reason is visible in the relation itself; reaching the critical value requires a limiting case, and going below it requires the charging law to be imposed rather than found. The circuit is a constructed instance and cannot show that the class is non-empty in nature, but it fixes the exponent by geometry rather than by assumption, and it permits the estimator used throughout to be calibrated against a system whose exponent is known.

One difficulty cuts across all of this and is only partly resolved. The equation describes a quantity that flows, while every realization in this paper transfers in units: charges, collisions, slips. **The exponent is then a property of a window containing many transfers rather than of any instant**, and the question of when it changes has no answer. **A reformulation in terms of the closing gap rather than the diverging observable settles part of this:** it gives the effect of a discrete transfer exactly, shows that a quantum of fixed size has an effect scaling as the square of the observable, and locates the truncation at the scale of the quantum itself, which two independent arguments place at the same value. **What it does**

not give is when the transfers arrive, since the rate at which a level receives them is set by the level above, and that question reopens at every level and has no answer at the fastest.

Two further results follow from the same accounting, and they concern how a cascade behaves under perturbation. Where a completing instance delivers everything it holds rather than a designated amount, the transfer scales with the gap, the number of events remaining before closure is constant at every scale, and the difficulty above does not arise at all; **such a cascade reaches an exponent of 2 without being set to it**, which is the one value a driver of bounded capacity can sustain. **Where instead the transfer is of fixed size, the state passed down is the remainder of the gap on division by the quantum**, and that operation is not invertible: each level discards which multiple of the quantum it contained, and a disturbance too small to measure determines the terminal state of a chain of any depth. **The two couplings therefore differ in kind**, one absorbing perturbations geometrically and the other amplifying them, and which a system does is a question about its transfer rather than its dynamics. **Nothing here is a loss of any physical symmetry**: time-translation invariance holds throughout, and what is lost is information.

No prediction is offered for any system, and the conjectures are labelled as such, in particular whether the fluctuations can be generated by nesting rather than imposed, and whether any real system satisfies the conditions. The framework is put forward as a descriptive instrument for recognizing intermittent superlinearity, together with the observations that would exclude it.

The Geometry of Bitcoin: A Single Observable and What It Cannot Settle

1. THE QUESTION AND ITS SCOPE

Standard growth descriptions of Bitcoin's price history, particularly those built on permanent diminishing returns, contain no superlinear, self-reinforcing term, and they are typically recalibrated as new market eras arrive. This paper asks a narrower question: **whether a self-reinforcing feedback loop can be detected at all**, using Bitcoin's absolute, fixed supply cap of 21 million coins as a reference point. To this end, it compares a tangent-based model, which contains a superlinear feedback term, with the best possible fit of the standard **Power Law** on the same historical data. The Power Law is the mathematical formulation of the diminishing-returns principle, in which price is assumed to grow as a fixed power of time, $P(x) = A \cdot x^n$. From this comparison, the paper then estimates an effective feedback exponent for the observed stretch (Section 3.3.1). The geometric divergence between the two models is documented statistically in Section 6, not merely asserted here.

A note on terminology: the title refers to the broader family this model belongs to. **Any system governed by superlinear feedback produces a finite-time singularity** (Section 3.3), and **the tangent function is the member of that family that admits a closed-form solution**. The word "feedback" is deliberate: the Power Law also grows superlinearly in time, but the feedback it implies is *sublinear* (exponent $1 - 1/n \approx 0.82$), whereas the tangent-based model adds a *superlinear* feedback term on top of it (exponent $1 + 1/n \approx 1.18$; Section 3.3). Throughout this paper, **"tangent-based model" refers to this specific closed-form instance**, while the general thesis concerns the superlinear-feedback family as a whole. Section 2.1 shows that this structure does not even require fixed exponents: exponents that fluctuate around the boundary value, or somewhat below it, can produce it as well.

It is worth being precise, however, about where the emphasis of this paper ultimately falls. Bitcoin's fixed supply is not, by itself, advanced here as a sufficient explanation for the geometric divergence documented in Section 6. As Sections 3.1 and 3.2 show, neither Bitcoin's own supply-side contraction nor ordinary demand growth, examined on their own terms, is capable of hosting the kind of self-reinforcing feedback loop that divergence requires — each produces only a mild, feedback-free, logarithmic structure. **The process this paper ultimately points to as the more likely host of that feedback loop is therefore external to Bitcoin's own supply dynamics**. Scarcity's role is best understood as a **necessary precondition** rather than the complete mechanism: it is what allows an asset to register the full effect of that external process undamped, in the absence of any elastic supply response that would otherwise absorb it. This distinction is developed across Sections 3.1 and 3.2 and revisited in Section 8.

Scope of this paper: This is a descriptive, exploratory measurement framework built on Bitcoin's historical price data. It is not a price-prediction tool, not financial advice, and not a claim of physical or economic law. **It does not forecast a price, a date, or any market outcome**, including at the mathematical asymptote described in Section 5. Its parameters are empirically calibrated on a specific historical stretch, are not independently derived, and are subject to the limitations discussed in Section 7. The singularity date implied by these parameters is a property of the calibrated curve, not a prediction. **Ultimately, as Section 8 discusses, this paper's most durable contribution may lie less in the specific price curve than in proposing this framework as a candidate instrument for observing the structural dynamics of the external system referenced in Sections 3.1–3.2. This broader framing is offered as a direction, not as a conclusively established claim.**

2. THE MATHEMATICAL MODEL AND EARLY-STAGE EQUIVALENCE

$$f(x) = A \left(\frac{\tan(B \cdot x)}{B} \right)^n$$

Where:

x = Time (e.g., Days since Genesis)

$f(x)$ = Valuation (Price)

A, B, n = Structural system parameters

This form originated as $\sqrt{\tan(x)}$, the case $A = B = 1, n = 1/2$; the specific parameters used in this paper are the result of calibration, not of the original construction.

A critical structural property of this tangent-based model is its relationship to conventional descriptions of diminishing returns, such as standard power law frameworks. In the network's early history, the two models are not merely similar—they are mathematically equivalent. This underlying behavior can be demonstrated using the Taylor series expansion for the tangent function.

For small values of x , the inner function expands as:

$$\frac{\tan(B \cdot x)}{B} \approx x + \frac{B^2 \cdot x^3}{3}$$

Assuming the parameter B is a sufficiently small positive constant, the squared term B^2 becomes microscopic. During the initial phases of the network's history (when x is relatively small), the cubic term remains virtually negligible. Consequently, the inner function effectively collapses to x , simplifying the overarching equation to:

$$f(x) \approx A \cdot x^n$$

This mathematical reality clarifies why standard power law formulations fit early historical nodes so effectively: **the tangent-based model mathematically contains the power law as its early-stage linear approximation**. Curve-fitting algorithms inherently struggle to isolate the B parameter using only early-stage data because the geometric curvature—the phase transition driving toward the asymptotic boundary—has not yet manifested measurably within the system's noise.

2.1 A Noise-Driven Formulation Without Fitted Exponents

The function above can also be written as the solution of a simple growth equation. Let $y(x)$ be a dimensionless state variable satisfying

$$\frac{dy}{dx} = k(1 + y^m), \quad y(0) = 0, \quad k > 0$$

and let it be mapped to the observed valuation by

$$f(x) = A \left(\frac{y(x)}{B} \right)^n$$

For $m = 2$ and $k = B$, the solution is $y(x) = \tan(B \cdot x)$, and the mapping reproduces the function above exactly. **For a constant exponent, m determines the character of the feedback.** A finite-time asymptote exists exactly when the integral $\int dy / (1 + y^m)$ converges, which holds if and only if $m > 1$. For $m < 1$, the feedback is **sublinear**: for large y , the state grows as a power of time, $y \propto x^{1/(1-m)}$, so f follows a flattening, power-law-type path with no asymptote. For $m > 1$, the feedback is **superlinear**, and y diverges at a finite point. The value $m = 1$ is the boundary between the two: the solution $y(x) = e^{kx} - 1$ grows exponentially but remains finite for every finite x .

The early-stage equivalence holds for every value of m . While $y \leq 1$, the feedback term satisfies $0 \leq y^m \leq 1$ for any $m \geq 0$, so the growth rate stays between k and $2k$. The state therefore grows at a nearly constant rate, and f keeps a power-law-type shape in the early phase whatever the character of the feedback. The feedback exponent only becomes visible once the state has grown large enough for the term y^m to dominate.

Replacing the fixed exponent with noise. Instead of a fitted value, let the feedback exponent fluctuate around a mean value at or below the boundary, while the mapping exponent is held fixed:

$$m(x) = \bar{m} + \varepsilon_m(x), \quad \bar{m} \leq 1 \quad n = 1$$

Here ε_m is a zero-mean noise term, bounded so that the exponent stays within $[0, 2]$. The noise is described by two properties: its **amplitude**, meaning how far it deviates from zero, and its **persistence**, meaning how long a deviation keeps the same sign. **The mapping exponent carries no noise.** With $n = 1$ the observed quantity is simply proportional to the state, so the whole of the stochastic behaviour enters through the feedback exponent m . The equation is solved step by step, with the exponent held constant within each step. With $\bar{m} = 1$, the average feedback sits exactly at the boundary; with $\bar{m} < 1$, **the average feedback is sublinear.**

Without noise, there is no asymptote. With the exponents held at their mean values, the equation is the constant-exponent case above. For $\bar{m} = 1$, the solution grows exponentially; for $\bar{m} < 1$, it grows only as a power law. In both cases it remains finite at every point, and in the early phase it reduces to a power-law-type shape.

With noise, the trajectory alternates between the two kinds of feedback. Whenever $m(x) < 1$, the path locally takes the flattening, power-law-type form. Whenever $m(x) > 1$, it locally takes the accelerating, tangent-type form. **Even when the average feedback is sublinear, the noise supplies superlinear stretches.** Because $dy/dx \geq k > 0$ at every point, **the state of this single equation never decreases:** the gains made during superlinear stretches are kept, while sublinear stretches only slow the growth down. **This applies to the latent state of one instance, not to an observed price series,** whose declines arise between nested instances of the equation rather than within one (Section 7.6.6). As the state grows, a superlinear stretch also needs less and less time to complete an asymptote. For a constant exponent $m > 1$ starting from a level y , the remaining time is at most

$$\frac{y^{1-m}}{k(m-1)}$$

which shrinks toward zero as y grows. Sufficiently persistent noise therefore eventually produces a superlinear stretch that is long enough, relative to the current level, to reach the asymptote, **whether the mean exponent lies at the boundary or below it.** This does not contradict the constant-exponent criterion. That criterion concerns a fixed exponent; with a fluctuating exponent, what decides the outcome is not the mean but the excursions above the boundary.

Numerical simulations of this construction support this. With a mean feedback exponent of exactly 1, **every run with noise reached a finite-time asymptote.** The median time to the asymptote was about 70 420 days with fast, weak noise, and about 8 850 days with slower, stronger noise. With a sublinear mean exponent of 0.96, **all 200 runs still reached an asymptote**, with a median of about 11 290 days and a 5th–95th percentile range of about 5 390 to 22 260 days. With mean exponents of 0.90 and 0.80, every tested run also reached an asymptote. A lower mean exponent delayed the asymptote, while more persistent noise brought it forward: at a mean of 0.80, the median time ranged from about 12 180 days with the most persistent noise tested to about 171 270 days with the least persistent.

Both pictures can therefore appear within a given observation window. The same equation can produce a trajectory whose asymptote lies beyond the window. Within the window, such a trajectory shows only the flattening, logarithmic-looking, power-law-type shape. With a mean exponent of 0.96 and only fast, weak noise, for example, just 7 of 20 simulated runs reached an asymptote within 400 000 days. The same equation can equally produce a trajectory that reaches its asymptote within the window, driven by the superlinear stretches that the noise supplies. Which of the two is observed depends mainly on the amplitude and persistence of the noise, and only secondarily on the mean exponent.

This is the central point of the formulation. The structure described by the function above, with an early resemblance to a power law and a possible finite-time asymptote, does not require specific fitted exponents, nor even an average feedback that is superlinear. It can arise from exponents whose average sits at the boundary value 1, or somewhat below it, once a sufficient degree of persistent noise is present. The constants A , B and n are therefore better read as a description of one observed trajectory than as defining constants of the underlying process. This is a property of the mathematical construction. No claim is made here about what the noise terms represent.

*** Note on the mapping exponent:** The value $n = 1$ is used here primarily to demonstrate structural similarity and to simplify the mathematical framework of the second part of this work. For Bitcoin specifically, a value greater than 1 would more accurately reflect reality. **The choice of n does not decide whether an asymptote exists.** The mapping is monotone, so it can neither create nor remove a singularity, and the criterion $m > 1$ is unchanged. What n alters is how the feedback appears in the observed valuation: the effective exponent seen there is $E = 1 + (m - 1)/n$, so a value $n > 1$ makes the observed feedback look closer to the boundary than the feedback acting on the state itself.

*** Note on the roles of the parameters:** The emphasis on m in this section is deliberate but incomplete. **m decides whether an asymptote can be reached, but it does not act alone.** The rate k is the internal clock of the system: it rescales the horizontal axis without changing the shape of the trajectory, and therefore sets *when* an asymptote is reached. It also sets the scale against which the noise is measured. The persistence of the noise is meaningful only relative to this clock, because the superlinear stretch required to complete an asymptote, $y^{1-m}/[k(m-1)]$, is inversely proportional to k . **The same noise can therefore be sufficient at one rate and insufficient at another**, and the times in days quoted above hold for the rate used in these simulations, not in general. A and n belong to the observation and affect neither the existence nor the location of the asymptote. In short, **m decides whether, k decides when, and the mapping decides how it appears. The exact relations between the parameters, and the precise conditions under which fluctuations of m force an asymptote, are set out in the second part of this work (Sections 2 and 3),** which adopts the normalization $B = k$, so that the observed quantity reduces to $A \cdot x$ in the early phase.

2.2 Comparison of Trajectories

The equivalence described above is easier to assert than to picture, and the simulation below is included so that it can be seen rather than taken on trust. **Three trajectories are drawn together:** the tangent-based function with the parameters calibrated in Section 6, the best-fitting Power Law with the exponent reported there, and one realization of the noise-driven formulation of Section 2.1, in which the feedback exponent is drawn at each step rather than fixed.

Environment. The simulation was run on Fedora Workstation under JupyterLab, in Python 3, using NumPy for the state array and the exponent draws and Matplotlib for the figure. It requires no other dependency and runs in a few seconds.

What the figure shows. Over the early part of the range the three curves are indistinguishable at the resolution of the plot, and they remain so well beyond the span of the historical data on which the parameters of Section 6 were calibrated. **They separate only afterwards, and they separate in three different ways:** the tangent-based curve reaches its asymptote and ends; the Power Law continues to flatten and never ends; the noise-driven trajectory follows neither, running above the Power Law for a long stretch before reaching an asymptote of its own, far later than the deterministic one. **No part of this behaviour was fitted.** All three begin from the same origin with the same rate, and the differences arise entirely from the feedback exponent.

The exponent of the third curve averages 0.80, below the critical value. Its draws are 1.5 and 0.5, taken with probabilities 0.3 and 0.7, so for most of the run the feedback is subcritical and the trajectory is being slowed rather than driven. **It reaches an asymptote regardless,** which is the result stated in Section 2.1 and proved in Part II, Section 3: what decides the outcome is not the average of the exponent but the recurrence of excursions above the boundary. The figure is one realization of that statement rather than evidence for it.

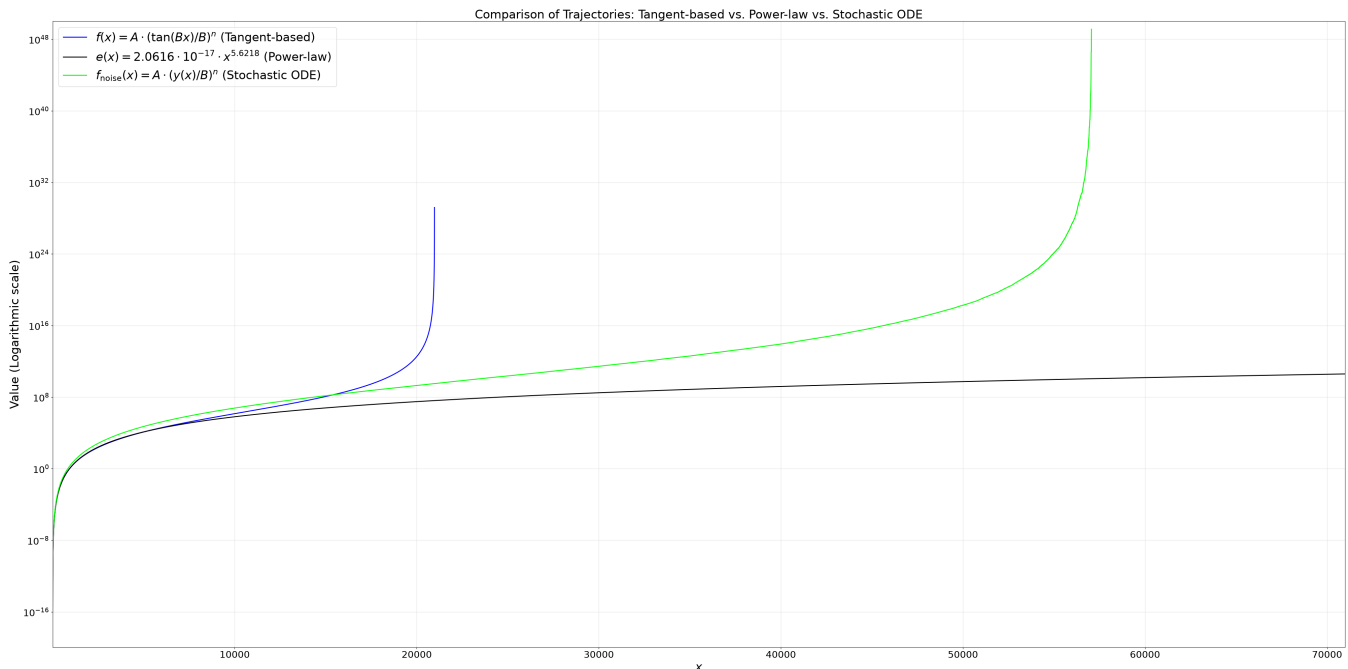


Figure 0a.

The point is the agreement, not the divergence. Section 2 argues that while the state remains small the feedback term is bounded between fixed limits whatever the exponent, so the early trajectory carries almost no information about it. The figure is that argument drawn: **an observer confined to the early range could not distinguish a curve that will terminate from one that never will, nor either from one whose exponent is fluctuating and whose mean lies below the boundary.** What separates them lies beyond the range in which they were observed.

Three cautions about reading it. First, **the green trajectory differs on every run**, since its exponent is drawn at random; the figure shows one realization, and the position of its asymptote in particular is not reproducible, varying widely from run to run. Second, the tangent-based curve is truncated at the last sampled point before its asymptote, which is an artifact of the step size and not a feature of the function. Third, the vertical axis spans many orders of magnitude, which compresses differences that would be large on any linear scale: **agreement to the eye on a logarithmic plot is agreement in order of magnitude and nothing finer**, which is why the quantitative comparison in Section 6 is made with log-residuals rather than by inspection.

All parameters are exposed and the figure is meant to be varied. The two exponent values and the probability between them, the rate, the mapping constants, the cutoff and the plotted range can all be changed. Lowering the probability of the supercritical draw delays the asymptote without removing it; removing the noise entirely and fixing the exponent at 2 reproduces the tangent-based curve exactly, since that is the case from which it was derived. **Changing one parameter and watching which part of the figure moves is more instructive than the figure itself**, which is why the code is given in full.

```

import numpy as np
import matplotlib.pyplot as plt

# --- 1. PARAMETERS AND CONSTANTS ---
A = 5.8535e-17
B = 74893 / 1e9          # ~ 7.4893e-5
n = 54666 / 10000        # 5.4666

# Global threshold cutoff for all functions
plotMaxThreshold = 1e50

# Theoretical asymptote position for the tangent argument ( $B \cdot x = \pi/2$ )
xAsymptote = np.pi / (2 * B) # ~ 20973.8

# Observation window
numSteps = int(np.ceil(xAsymptote)) + 50000
x = np.arange(numSteps, dtype=float)

# --- 2. CLOSED-FORM ANALYTICAL EVALUATION ---
# f(x): Tangent-based function
arg = B * x
with np.errstate(divide='ignore', invalid='ignore'):
    # Valid only up to the first singularity ( $0 \leq B \cdot x < \pi/2$ )
    tanVal = np.where(arg < np.pi / 2, np.tan(arg), np.nan)
    fx = A * np.power(tanVal / B, n)

# e(x): Power-law approximation
ex = 2.0616e-17 * np.power(x, 5.6218)

# Apply global 1e50 cutoff to analytical curves
fx = np.where(fx > plotMaxThreshold, np.nan, fx)
ex = np.where(ex > plotMaxThreshold, np.nan, ex)

# --- 3. STOCHASTIC ORDINARY DIFFERENTIAL EQUATION (Simulated y(x)) ---
#  $dy/dx = k * (1 + y^m)$ ,  $y(0) = 0$ , where  $k = B$  and  $m = 2$  yield the deterministic tangent
k = B
y = np.zeros(numSteps)
y[0] = 0.0

fxSim = np.zeros(numSteps)
fxSim[0] = 0.0

# Noise generation for exponent m
p = 0.3
mNoise = np.random.choice([1.5, 0.5], size=numSteps, p=[p, 1 - p])

terminated = False

for i in range(1, numSteps):
    if terminated:
        fxSim[i] = np.nan
        continue

    # Exponent from previous step's noise realization
    m = mNoise[i - 1]

    # Step size  $dx = 1$ 
    dy = k * (1.0 + y[i - 1]**m)
    yTemp = y[i - 1] + dy

    # Calculate observable  $f_{\text{noise}}$  value for this step
    with np.errstate(divide='ignore', invalid='ignore', over='ignore'):
        currentFxSim = A * ((yTemp / B) ** n)

    # Permanent singularity termination directly checked against plotMaxThreshold (1e50)
    if currentFxSim > plotMaxThreshold or np.isnan(currentFxSim) or np.isinf(currentFxSim):
        fxSim[i:] = np.nan
        terminated = True
    else:
        y[i] = yTemp
        fxSim[i] = currentFxSim

# --- 4. VISUALIZATION ---
plt.figure(figsize=(40, 20))

```

```

# Plot from x = 1 since x = 0 is undefined on a logarithmic scale
plotStart = 1
plotEnd = numSteps

plt.plot(x[plotStart:plotEnd], fx[plotStart:plotEnd],
        color='blue', linewidth=2, alpha=1.0,
        label=r'$f(x) = A \cdot (\tan(Bx)/B)^n$ (Tangent-based)')

plt.plot(x[plotStart:plotEnd], ex[plotStart:plotEnd],
        color='black', linewidth=2, alpha=1.0,
        label=r'$e(x) = 2.0616 \cdot 10^{-17} \cdot x^{5.6218}$ (Power-law)')

plt.plot(x[plotStart:plotEnd], fxSim[plotStart:plotEnd],
        color='lime', linewidth=2, alpha=1.0,
        label=r'$f_{\mathrm{noise}}(x) = A \cdot (y(x)/B)^n$ (Stochastic ODE)')

plt.yscale('log')
plt.title('Comparison of Trajectories: Tangent-based vs. Power-law vs. Stochastic ODE', fontsize=25)
plt.xlabel('$x$', fontsize=25)
plt.ylabel('Value (Logarithmic scale)', fontsize=25)
plt.xlim(plotStart, plotEnd)
plt.ylim(top=plotMaxThreshold)
plt.tick_params(axis='both', which='major', labelsize=20)
plt.legend(loc='upper left', fontsize=25)
plt.grid(True, which='both', alpha=0.3)

plt.tight_layout()
plt.show()

```

JupyterLab code for Figure 0.

3. THE THEORETICAL CORE: SYSTEM DYNAMICS AND STRUCTURAL REFLEXIVITY

To move beyond pure curve-fitting, this section proposes one possible structural motivation for the valuation function from a small set of economic assumptions. **The approach taken here is an attempt, not a demonstration.** Whether a self-reinforcing feedback loop actually exists cannot be observed directly, so this paper tries to detect it indirectly. **A tangent-based model, which contains a superlinear feedback term, is compared with a baseline without such a term, the Power Law** (Section 6.3), and **the effective feedback exponent is then estimated from the fitted model** (Section 3.3.1). A closer fit by the tangent-based model is consistent with the presence of such a loop, but does not by itself prove it. Because the same qualitative structure can also arise from fluctuating exponents (Section 2.1), the estimated exponent is best read as an effective value, not as a fixed constant of the process. The argument proceeds in three steps. **Section 3.1 first establishes a negative result: a contraction of Bitcoin's liquid supply cannot, on its own, produce the geometry documented in Section 6. Section 3.2 extends this exclusion to ordinary demand growth. Section 3.3 then introduces the minimal additional ingredient that can — a superlinear feedback term — and shows that it maps exactly onto the empirical function of Section 2.**

The structural question this section addresses is narrower than it may first appear. **The distinction documented in Section 6 — between a feedback-free structure and the superlinear, feedback-driven structure of the tangent-based model — is not a claim about whether price goes up or down.** It is a claim about which side of the price equation, if either, contains a self-reinforcing loop at all: the perpetually expanding external capital supply, or the strictly fixed, inelastic supply of the Bitcoin network, set by protocol at 21 million units as a matter of definition rather than observation. **Zero elasticity removes the dampening that an elastic production response would otherwise apply, but removing a dampening force is not the same as introducing a self-reinforcing one.** A fixed or contracting supply therefore does not, by itself, force any particular price outcome, and the available data show no mechanical link between the two (see the note below). This is a statement about the mathematical structure of the supply side, not a forecast of how Bitcoin's liquid supply will actually evolve, and it holds whatever those dynamics turn out to be. **At least one additional process, such as the reflexivity dynamic of Section 3.3, is needed to connect this structural condition to accelerating price behavior.** If a feedback loop is present in the data at all, it most plausibly operates on the demand side: not because rising valuation is assumed in advance, but because Sections 3.1 and 3.2 exclude the alternatives.

* This absence of a mechanical link is visible in the available exchange-reserve data for the roughly 2.5-year period for which it could be obtained. Before March 2024, reserve and price rose together. From March 2024 onward, however, reserves entered a steady declining trend that continued largely unchanged through both the subsequent 2024 bull market and the bear-market decline that followed it — the same downward reserve trend persisted regardless of price direction. This pattern is consistent with the simplest, structure-free depletion case, $L(x) = L_0 - cx$, used as the baseline assumption in Section 3.1, rather than with a supply response that reacts to price. Under the strict reading of the differential equation in Sections 3.1–3.2, a supply trend of this kind does not, by itself, determine the sign of dP/dx without also knowing $F(x)$ — consistent with, not contrary to, the argument developed there. This is offered as a qualitative, visually-observed illustration from a specific historical window, not a statistical test of the broader claim.

3.1 A Structural Lower Bound: Why Supply Contraction Alone Is Insufficient

To test whether Bitcoin's fixed-supply property is, by itself, capable of hosting a feedback loop of the kind this paper's model requires, consider a simple market-impact framework, in which the rate of price change is proportional to external demand and inversely proportional to available liquid supply:

$$\frac{dP}{dx} = k \cdot \frac{F(x)}{L(x)}$$

where $F(x)$ represents external capital inflow and $L(x)$ represents available liquid supply. **As the simplest possible case — one requiring no additional structure or feedback of its own — consider a supply that depletes linearly** toward a finite exhaustion point x^* :

$$L(x) = L_0 - c \cdot x, \quad x^* = L_0/c$$

Assuming, for simplicity, a constant external demand $F(x) = F_0$, the differential equation can be solved directly by separation of variables:

$$P(x) = P_0 + \frac{k F_0}{c} \cdot \ln \left(\frac{L_0}{L_0 - cx} \right)$$

As $x \rightarrow x^*$, the denominator $L_0 - cx \rightarrow 0$, and $P(x)$ does indeed diverge — but only **logarithmically**. **This is a structurally feedback-free divergence**: each tenfold reduction in the remaining liquid supply adds only a fixed increment, $(k F_0/c) \cdot \ln 10$, to price, regardless of how large price itself has already grown — there is no term through which price's own level feeds back into its future rate of change. Even at 99.99999% of the way to the exhaustion point x^* , the logarithmic factor has reached only about 16. **A linear, constant-rate contraction of supply is therefore mathematically incapable, on its own, of introducing the self-reinforcing structure this paper's model requires** — independently of whether Bitcoin's real liquid supply actually behaves this way, this specific candidate mechanism does not contain a feedback loop.

This conclusion is robust to the exact shape of $L(x)$ near x^* : in general, if liquid supply approaches zero as $L(x) \propto (x^* - x)^r$, integration shows that $P(x)$ diverges as a genuine power law only when $r > 1$ — that is, only if the depletion itself *accelerates* as the exhaustion point is approached, which is itself a feedback-like property of the supply function, not a consequence of price. The case $r = 1$, the simplest, structure-free depletion case considered above, sits exactly at the boundary between a power-law blow-up and no blow-up at all, and produces only the weak logarithmic case derived above.

The same framework also quantifies how demanding a supply-side explanation would have to be. Near its asymptote $x^* = \pi/(2B)$, the tangent satisfies $\tan(B \cdot x) \approx 1/(B(x^* - x))$, so the empirical function of Section 2 diverges as:

$$f(x) \propto (x^* - x)^{-n}$$

Under constant external demand, reproducing this divergence from supply contraction alone would require $dP/dx \propto (x^* - x)^{-(n+1)}$, that is, a liquid supply vanishing as:

$$L(x) \propto (x^* - x)^{n+1}$$

With the published $n = 5.4666$, this corresponds to a depletion exponent of $r \approx 6.5$, compared with $r = 1$ for the simplest, structure-free case considered above. The gap is qualitative rather than marginal: reproducing the model's structure from supply alone would require the depletion function itself to already contain a sharply accelerating, self-reinforcing shape — which is precisely the kind of structural feedback this section set out to test for, not something a plain, unadorned depletion process supplies for free.

A useful structural inference follows from this: a supply side that depletes without any additional, independently-motivated accelerating structure of its own cannot host the feedback loop this model's geometry requires. If such a loop is present in the observed data at all, it must therefore originate predominantly in the external capital-inflow term, $F(x)$, rather than in Bitcoin's own supply-side mechanics. This paper does not take a position on the nature or origin of that external term — doing so falls outside its scope, which is limited to Bitcoin's own price geometry — and treats it here strictly as a structural placeholder required to close the mathematical picture.

The reflexivity term introduced in Section 3.3 is the simplest shape this placeholder can take. If the state variable is written in dimensionless form y , and inflow grows with it as $F = F_0 \cdot (1 + y^2)$ while liquid supply remains approximately constant at L_0 over the relevant interval, the market-impact equation above reduces exactly to:

$$\frac{dy}{dx} = \frac{k F_0}{L_0} \cdot (1 + y^2)$$

This constrains only the *shape* of the external term, not its nature. **The same market-impact equation therefore contains both cases side by side:** a feedback-free demand term against a linearly depleting supply yields the logarithmic, non-reinforcing structure derived above, while a feedback-carrying demand term against a finite supply yields the self-reinforcing structure developed in Section 3.3 — the same equation can host either, depending entirely on which side, if any, contains the loop.

3.2 A Second Structural Exclusion: Nominal Demand Growth Does Not Change the Order of the Singularity

A natural extension of the argument in Section 3.1 is to test whether a steadily *growing*, rather than strictly constant, external demand term could itself introduce the missing feedback structure, without requiring the reflexivity term proposed in Section 3.3. This can be tested within the same market-impact framework used previously.

In place of the constant demand assumed in Section 3.1, suppose demand grows exponentially at some constant rate i — a steadily growing, but still structurally feedback-free, term:

$$F(x) = F_0 \cdot e^{ix}$$

Combined with the same linear supply contraction established in Section 3.1, $L(x) = L_0 - cx$, the governing equation becomes:

$$\frac{dP}{dx} = k \cdot \frac{F_0 \cdot e^{ix}}{L_0 - cx}$$

As $x \rightarrow x^* = L_0/c$, the exponential term e^{ix} approaches a finite, well-behaved value — it is the supply term, $1/(x^* - x)$, that governs the order of the singularity. Numerical integration confirms this directly: for any rate i , the increase in $P(x)$ across each successive order-of-magnitude approach to x^* stabilizes at a constant value — the defining signature of a logarithmic, feedback-free divergence — with i affecting only the overall scale of the curve, not its underlying structure. Even at $i = 0.20$, an extreme, arbitrarily-chosen growth rate, the divergence remains logarithmic in order.

This allows a specific structural possibility to be excluded: **a linearly contracting supply combined with any merely exponential growth in demand, however rapid, cannot by itself introduce anything beyond the weak, feedback-free divergence already derived in Section 3.1.** Neither Bitcoin's supply contraction alone, nor its combination with a steadily growing but structurally passive demand term, is capable of hosting the self-reinforcing loop this paper's model requires. This reinforces the point stated in Section 1: whatever process is ultimately responsible for that structure is external to Bitcoin's own supply dynamics, and is not reducible to ordinary, non-reinforcing demand growth either.

3.3 The Reflexivity Term and Its Exact Link to the Empirical Function

We can model this feedback directly using differential calculus. The rate of change of the system's state with respect to time (dy/dx) is driven by two distinct forces: a constant baseline representing continuous external inflow, and a reflexivity term proportional to y^2 , **representing price-dependent inflow that compounds as valuation rises**. Here y is a **dimensionless state variable**; its exact relationship to price in USD is given below.

Mathematically, this relationship is expressed by the differential equation:

$$\frac{dy}{dx} = k \cdot (1 + y^2)$$

Where k is a proportionality constant reflecting the structural pressures of the system.

In calculus, the fundamental analytical solution to this specific non-linear differential equation is the tangent function:

$$y(x) = \tan(k \cdot x + C)$$

*** Note on Structural Derivation vs. Empirical Parameters:** The relationship between this solution and the empirical function $f(x) = A(\tan(Bx)/B)^n$ of Section 2 is **exact rather than merely qualitative**. The Genesis condition $y(0) = 0$ fixes $C = 0$, and identifying $k = B$ gives $y(x) = \tan(B \cdot x)$. The empirical function is then a monotone power transform of this solution:

$$f(x) = A \cdot \left(\frac{y(x)}{B} \right)^n$$

The reflexivity equation therefore governs the dimensionless state variable y , while A and n parameterize the mapping from y to price in USD; **dividing by B normalizes the initial slope, so that $f(x) \approx A \cdot x^n$ in the early phase** (Section 2). Expressed directly in price terms, f satisfies:

$$\frac{df}{dx} = n \cdot A^{1/n} \cdot f^{1-1/n} + n \cdot B^2 \cdot A^{-1/n} \cdot f^{1+1/n}$$

The first term alone yields exactly the Power Law $A \cdot x^n$: the Power Law is precisely the trajectory this system would follow in the absence of feedback. The second term is a superlinear feedback term with an **effective exponent of $1 + 1/n \approx 1.18$** at the published $n = 5.4666$; because this exponent exceeds 1, it produces a finite-time singularity at $x^* = \pi/(2B) \approx \text{day } 20,974$. This date is a property of the calibrated curve of Section 6.2, not a forecast; as Section 7.1 shows, it is highly sensitive to the calibration.

Why a superlinear term is needed at all is established in Sections 3.1 and 3.2, not here: reproducing the fitted curve from supply contraction alone would require a depletion exponent of $r \approx 6.5$, against $r = 1$ for the simplest, structure-free case. This section shows only that a minimal feedback term of this kind is **sufficient** to generate the observed geometry — the tangent-based curve tracking marginally flatter than the Power Law in the network's early history, then measurably steeper in its later history, as confirmed by the MACE comparison in Section 6.3 and directly visible in the Pine Script visualization of Section 6.5 — and makes its mapping onto the empirical function explicit. **It does not derive the numerical values of A , B and n , or the choice of the quadratic form over other superlinear closed forms, from first principles.**

It is worth noting that the emergence of a finite-time singularity is not narrowly dependent on the specific quadratic form of the reflexivity term. Any superlinear feedback assumption ($y' = k(1+y^m)$) for $m > 1$ produces the same qualitative outcome — a mathematical collapse of the time axis. The quadratic case ($m=2$) is adopted here specifically because it yields a closed-form, empirically tractable solution — the tangent function — rather than because market structure is assumed to dictate this exact exponent.

This choice is not arbitrary in an even stronger sense, however: it was not selected in isolation and then merely hoped to fit the data. The tangent-based functional form was independently supported in Section 6 through direct empirical curve-fitting against historical price nodes, where it demonstrated a measurably tighter fit (lower Mean Absolute Centered Error) than the best possible Power Law calibrated on the same data. The quadratic form should not, however, be read as a claim that price feeds back on itself with exponent 2. **Through the transform $f = A(y/B)^n$, the effective feedback exponent in price terms is $1 + 1/n \approx 1.18$;** the value 2 in the latent equation is the choice that yields a closed-form solution, while the empirically fitted n maps to this effective exponent — a figure whose precise value is only as reliable as the specific historical data used to derive it, as Section 7.1 illustrates. This paper does not, and cannot, assess whether this effective exponent will continue to hold going forward; only that its current value rests on a fitted, not independently derived, quantity. Section 6's fit comparison supports this functional form empirically, but that is a statistical finding, not a derivation of the exponent from market structure.

*** Note on Market Reflexivity:** In traditional finance, the concept of "reflexivity" was famously pioneered by George Soros to describe cognitive and behavioral feedback loops—where investor perception alters market fundamentals, which in turn validate the perception. However, the reflexivity modeled in this framework is strictly *structural and mathematical*, not psychological.

The two accounts are not unrelated, however. **The measurable divergence this paper documents between the tangent-based model and the Power Law (Section 6.3) is coherent with, rather than independent of, Soros's account:** a market whose participants act on reflexive perception — where the expectation of further gains draws in demand that helps validate the expectation — would be expected to leave behind exactly this kind of structural signature, a departure from the diminishing-returns baseline in the direction of self-reinforcing acceleration. This paper does not claim to have measured Soros's mechanism directly, or to have established it as the specific source of the effect documented here; it notes only that the structural direction of the statistical result and the qualitative prediction of reflexivity theory, arrived at independently, point the same way.

3.3.1 Exact Derivation of the Effective Feedback Exponent

To ensure complete structural transparency, the explicit transformation from the dimensionless state variable y to the price domain $f(x)$ is derived below. The base equations are the dimensionless reflexivity condition and the price mapping function:

$$\frac{dy}{dx} = B \cdot (1 + y^2) \quad \text{and} \quad f = A \cdot \left(\frac{y}{B} \right)^n$$

Applying the chain rule to differentiate f with respect to x gives:

$$\frac{df}{dx} = n \cdot A \cdot \left(\frac{y}{B} \right)^{n-1} \cdot \frac{1}{B} \cdot \frac{dy}{dx}$$

Substituting the reflexivity condition $B \cdot (1 + y^2)$ for dy/dx eliminates the denominator B :

$$\frac{df}{dx} = n \cdot A \cdot \left(\frac{y}{B} \right)^{n-1} \cdot (1 + y^2)$$

Distributing this across the terms in the parenthesis separates the baseline from the feedback mechanism:

$$\frac{df}{dx} = n \cdot A \cdot \left(\frac{y}{B} \right)^{n-1} + n \cdot A \cdot \left(\frac{y}{B} \right)^{n-1} \cdot y^2$$

To express the derivative purely in terms of price f , the initial mapping function is inverted to isolate the state variables:

$$\frac{y}{B} = \left(\frac{f}{A} \right)^{1/n} \quad \text{and} \quad y = B \cdot \left(\frac{f}{A} \right)^{1/n}$$

Substituting these inverted relations back into the first distributed term (the Power Law baseline) yields:

$$n \cdot A \cdot \left(\frac{f}{A} \right)^{(n-1)/n} = n \cdot A^{1/n} \cdot f^{1-1/n}$$

Substituting into the second term (the superlinear reflexivity) yields:

$$n \cdot A \cdot \left(\frac{f}{A} \right)^{1-1/n} \cdot B^2 \cdot \left(\frac{f}{A} \right)^{2/n} = n \cdot B^2 \cdot A^{-1/n} \cdot f^{1+1/n}$$

Recombining these components produces the final exact differential equation evaluated in price terms:

$$\frac{df}{dx} = n \cdot A^{1/n} \cdot f^{1-1/n} + n \cdot B^2 \cdot A^{-1/n} \cdot f^{1+1/n}$$

Using the empirically calibrated parameter $n = 5.4666$, the exponent of the feedback term evaluates to $1 + 1/5.4666 \approx 1.1829$. Because this exponent strictly exceeds 1, it shows that, for this calibration, the superlinear character of the feedback carries over from the dimensionless state space into the observed price domain.

3.4 A Structural Observation: The Biological Converse

From a purely mathematical perspective, it is worth noting that the differential equation proposed in this framework shares a precise structural relationship—an exact sign inversion—with classical biological models of population growth.

In biology, the standard logistic growth model introduces a negative feedback loop: as a population (y) expands, it consumes the finite resources of its environment, limited by a physical carrying capacity (K). Mathematically, this is expressed as:

$$\frac{dy}{dx} = r \cdot y \cdot \left(1 - \frac{y}{K} \right)$$

This negative feedback forces the growth rate to eventually approach zero, resulting in a horizontal asymptote (a flattening S-curve) where the population stabilizes below the carrying capacity K .

To construct a rigorous, rather than merely rhetorical, counterpart to this model, we can take the exact sign inversion of the logistic equation—flipping the restraining term into an accelerating one:

$$\frac{dy}{dx} = r \cdot y \cdot \left(1 + \frac{y}{K} \right)$$

Structurally, this is not merely an analogous or loosely inspired formulation—it is the **literal converse of the logistic equation**, produced by inverting the sign of the single term separating unconstrained growth from self-limiting growth. Where the logistic model's carrying capacity K acts as a dampening ceiling, this inverted structure transforms K into an accelerating pressure: as y grows, the very term that once suppressed further growth now compounds it. Solving this equation confirms that, like the reflexivity-based model introduced in Section 3.3, this exact converse of the logistic equation likewise produces a finite-time singularity rather than a stabilizing plateau.

This structural relationship is offered as an illustration, not as support for any particular mechanism. In the logistic model, the term containing K is what makes growth self-limiting; in its converse, the same term makes growth self-reinforcing. **The external capital-inflow term treated as a structural placeholder in Section 3.1 would play the role of this reinforcing term**, not named or identified further here. Rather than approaching a horizontal limit where growth flattens, the inversion drives the geometric curvature upward into a vertical asymptote. While the exact exponents and constants of Section 3.3's reflexivity term remain a modeling simplification (Sections 7.1 and 7.5), this sign-inverted counterpart to a well-established biological model shows that the qualitative behavior, a finite-time singularity, is not an artifact unique to this framework. **It follows from the presence of a self-reinforcing term, not merely from the absence of a dampening one**, which is the same distinction drawn in Section 3.

4. STRUCTURAL VALIDATION VIA TEMPORAL TRANSLATION (PHASE SHIFTING)

To rigorously test whether the derived asymptotic geometry represents a genuine systemic property rather than a mathematically overfitted baseline, we can apply a simple phase shift—a horizontal translation along the time axis. By shifting the original baseline function forward in time by a constant offset, we can examine how this translated geometry interacts with the market's upper cyclical extremes.

$$f_{shift}(x) = f(x + \Delta t)$$

Where Δt represents a constant positive time offset.

The evidentiary weight of this validation method relies on geometric alignment and statistical probability. If an arbitrarily chosen, translated mathematical function happens to intersect or bound a single cyclical peak, it proves absolutely nothing. Intersecting two distinct peaks could still be reasonably dismissed as a geometric coincidence.

However, the statistical significance scales non-linearly with each additional point of contact. If the exact same rigid curvature—shifted by a single, constant temporal offset—consistently aligns with or bounds the upper cyclical extremes across multiple, fundamentally distinct market eras, the probability of random coincidence drops drastically. If such a curve were to accurately map tens of thousands of data points or multiple macro-cycle limits, it would substantially shift the balance of probability away from coincidence and toward a structural mathematical rule.

While no model can claim absolute deterministic certainty, this temporal translation serves as a substantive structural test. When the overarching geometry naturally fits both the macro-bottoms (the baseline) and the macro-tops (the shifted boundary) **without requiring any alteration to the function's internal curvature**, it lends meaningful empirical support to the idea that tangent geometry plays a role in shaping the asset's macro-dynamics.

5. THE MATHEMATICAL ASYMPTOTE, MARGINAL PRICING, AND PHASE TRANSITION

By its fundamental nature, the tangent function possesses a structural boundary where it reaches a vertical asymptote. In a generalized, parameter-free form, this mathematical limit is defined as follows:

$$\lim_{x \rightarrow \pi / 2B} f(x) = \infty$$

Note: this is not a price forecast. A literal infinite price cannot exist in any real market. This limit describes a mathematical boundary of the function's domain, not a target valuation — see below for what it does describe.

The mathematical asymptote is read here instead as **a possible structural boundary**: a point at which, if the feedback loop described in Section 3 genuinely exists, the assumptions underlying continuous, liquid price discovery, most notably the presence of a willing marginal seller at any given price, may no longer hold in their current form.

What would happen at or beyond that point lies outside the scope of this paper. This paper does not examine what would follow if such a feedback loop genuinely exists and meets a shrinking pool of available liquid supply. The asymptote should therefore not be read as a forecast of any valuation, date, or economic outcome. The only claim advanced here is narrower and conditional: **if such a loop exists, the mechanism of marginal pricing may run into difficulties as available liquid supply approaches zero**, and the vertical asymptote of the tangent function is intended to represent this possible instability geometrically.

This macroscopic dynamic can be conceptually viewed through the lens of *econophysics*, drawing a structural analogy to statistical mechanics. As explored in the foundational work of researchers like R. N. Mantegna, H. E. Stanley, and Didier Sornette (who specifically modeled finite-time singularities in complex market systems), daily price fluctuations are highly chaotic—much like the unpredictable, microscopic collisions of gas molecules in a closed system. Yet, within this framework of extreme micro-volatility, the macroscopic trajectory of a system does not need to be strictly deterministic to carry a definite structural tendency. The noise-driven formulation of Section 2.1 illustrates this: fluctuating exponents can still produce an asymptotic structure, even though the timing of the asymptote remains highly variable.

Finally, it is essential to address the domain of the equation. Strictly mathematically, the tangent curve is periodic, repeating infinitely across the coordinate system. However, this economic model defines the Genesis Block as the absolute origin (zero time, zero value). Therefore, only the primary, initial branch of the tangent function holds any physical or chronological significance. Subsequent repeating branches of the function exist in mathematical theory but carry no valid meaning for the asset's real-world lifecycle.

6. HISTORICAL APPLICATION, VISUAL COMPARISON, AND PARAMETERIZATION

The parameterization below describes one specific historical stretch, not the process as a whole. It is calibrated on seven tangency nodes between Day 745 and Day 6388, and its validity does not extend automatically beyond that stretch. As Section 2.1 shows, the same qualitative structure can arise from fluctuating exponents without any fitted values. **Fixed parameters are used here out of necessity:** a quantitative estimate of the effective feedback exponent (Section 3.3.1) requires a concrete curve, fitted to a concrete stretch of data. The resulting exponent is therefore best read as **an effective value describing that stretch as a whole**, not as a constant of the underlying dynamic.

To transition from the theoretical framework to observed market action, the mathematical model must be mapped directly onto live market data. However, establishing this fit requires an objective methodology for defining historical "support nodes."

A note before proceeding: the specific numerical parameterization derived below is a statistical curve fit, not a claim of physical law, and it can be broken. Section 7.3 examines this directly, treating *this particular parameterized curve* as a measurement structure with a limited calibration range, distinct from the *underlying structural dynamic* it attempts to describe (Section 3). An extreme enough market event could end the validity of the parameters below, and of the effective exponent derived from them, without deciding anything about the broader thesis in either direction. Readers are encouraged to keep this distinction in mind throughout the following sections.

6.1 Methodology of Tangency and Constraint Satisfaction

A common pitfall in financial modeling is arbitrary curve-fitting—selecting random local lows simply because they visually align with a desired trajectory. To avoid this, our historical nodes are defined by a strict mathematical condition: **tangency**.

A point qualifies as a valid structural support node if it satisfies two simultaneous criteria on a logarithmic scale:

1. **Value Equivalence:** The actual price and the model's value are virtually identical.
2. **Slope Equivalence:** The rate of change (momentum or derivative) of the actual price matches the slope of the model's curve.

Visually, this means the mathematical curve does not awkwardly slice through the price action; instead, it smoothly "kisses" the lower boundary of the price data before separating. Because of this dual requirement, true mathematical touch points do not always land on the absolute lowest daily wick of a bear market; they often manifest as lateral contact points where the market's momentum perfectly aligns with the function's structural geometry.

The seven tangency nodes used to calibrate this model were themselves subjectively identified through visual inspection — this is not disputed. What is not subjective, however, is the outcome once those seven points are fixed: the resulting curve is heavily over-constrained, since it must simultaneously avoid ever exceeding price across multiple, independent multi-year market eras. This is a **constraint-satisfaction condition, not a free optimization** — arbitrary alternative candidate points (for example, the 2018 or 2022 bear-market bottoms) could not have been substituted without violating the non-crossing condition elsewhere in the historical record.

This can be tested directly. The 2018-12-15 bear market low (~\$3,122) and the 2022-11-21 post-FTX low (~\$15,479) were never used to calibrate any parameter of this model. Evaluating the currently published curve at these two dates shows it remains below both realized prices (by 37.6% and 11.0%, respectively) — the curve was never at risk of being crossed by data it had no knowledge of. This does not constitute proof of the model's validity, but it is a genuine, independently checkable consistency test: the seven-node calibration did not merely fit seven arbitrary points — **it produced a curve that also respects historical price action outside its own calibration set.**

6.2 Calibrated Parameters and Baseline Functions

Applying this tangency methodology across Bitcoin's macroeconomic history yields the following integrated system parameters for the tangent-based model. These constants describe the calibrated historical stretch closely, serving as a descriptive geometric fit rather than a guaranteed deterministic law.

$$f(x) = A \left(\frac{\tan(B \cdot x)}{B} \right)^n$$

Tangent Parameters:
x = Days since Genesis Block (2009-01-03)
f(x) = Price in USD
A = 5.8535 × 10⁻¹⁷
B = 7.4893 × 10⁻⁵
n = 5.4666

The Standard Power Law Model (Optimized Baseline):

To provide a rigorous, non-arbitrary comparison, the Power Law parameters provided below were independently derived via Ordinary Least Squares (OLS) log-log linear regression on the exact same seven tangency nodes used to calibrate the tangent-based model.

$$P(x) = A_{PL} \cdot x^{n_{PL}}$$

Optimal Power Law Parameters:
A_{PL} = 2.0616 × 10⁻¹⁷
n_{PL} = 5.6218

6.3 Statistical Divergence and Historical Comparison: Tangent-based model vs. Power Law

The comparative advantage of the tangent function model becomes visible when visualized alongside the optimized Power Law using the same historical dataset. Even when the Power Law is mathematically forced into its best possible fit (OLS regression), it struggles to capture the structural curvature of the asset over time. It systematically overprices the middle eras while missing the geometric acceleration of recent macro-bottoms.

Identified Tangency Nodes & Model Comparison:

Days (x)	Actual (USD)	Tangent f(x)	Tangent (Δ ln)	Power Law P(x)	Power Law (Δ ln)
745	0.313	0.296	-0.057	0.289	-0.080
1049	2.05	1.931	-0.060	1.978	-0.036
1242	4.98	4.882	-0.020	5.112	0.026
1324	7.05	6.940	-0.016	7.323	0.038
2425	198	197.97	-0.000	219.91	0.105
4087	3850	3848.87	-0.000	4136.64	0.072
6388	57735	57716.98	-0.000	50938.45	-0.125

Statistical Summary (Centered Residuals)

For explicit methodological transparency, the Mean Absolute Centered Error (MACE) referenced in the summary is calculated by first defining the individual logarithmic residuals (r_i), determining their arithmetic mean (μ) to isolate systemic bias, and then computing the average of the absolute deviations from this mean:

$$MACE = \frac{1}{N} \sum |r_i - \mu|$$

Where:
N = Total number of tangency nodes
r_i = ln(Model) - ln(Actual) for each given node
μ = The arithmetic mean of all r_i values

By design, the OLS Power Law regression centers its Mean Error at ~0.000. For a direct structural comparison, we isolate the shape accuracy of both curves by mean-centering their residuals (shifting the baseline to a true zero-error mean). The resulting **Mean Absolute Centered Error for the Power Law is ~0.069 (6.9%)**. In contrast, the tangent-based model achieves a **Mean Absolute Centered Error of approximately 0.021**

(2.1%). This suggests that the inherent geometry of the tangent function maps the observed macro-bottoms more than three times more accurately than the best mathematically possible linear log-log fit.

This asymmetry can be demonstrated directly. If the Power Law is instead forced to pass exactly through the two most recent nodes (Day 4087 and Day 6388) — the same nodes the tangent-based model tracks almost perfectly without being specifically forced to — its fit to the earlier historical nodes collapses, with log-residuals of roughly -0.9 to -0.5 (the curve running roughly 40–60% below the realized price) at Days 745 through 1324. The tangent-based model faces no equivalent trade-off: it achieves near-zero error at the recent nodes while remaining within 5-6% at the early ones, simultaneously. **This is a genuine structural asymmetry between the two functional families, not merely an artifact of the tangent model's additional free parameter** — a two-parameter Power Law cannot satisfy both epochs at once, regardless of how its parameters are chosen, while the tangent-based family can. This illustrates that **the two curves' early-stage resemblance is not evidence of equivalence, and the structural difference between them becomes visible precisely where the two curves are forced to account for the network's more recent history**. A visual comparison of the two curves against realized price action is available via the TradingView Pine Script referenced in Section 6.5.

*** Note on Error Measurement and R²:** Traditional metrics like the Coefficient of Determination (R²) are intentionally omitted here. In hyper-exponential datasets spanning multiple orders of magnitude, standard R² heavily overweights absolute errors at late-stage high valuations, allowing models to artificially inflate accuracy by overfitting recent data while ignoring massive percentage errors in early stages. Instead, accuracy is evaluated strictly using individual logarithmic residuals ($\ln(\text{Model}) - \ln(\text{Actual})$), providing a symmetric measure that weights all eras equally.

*** Note on Data Provenance:** To ensure consistency, historical pricing data from Day 1242 onward is sourced directly from the Bitstamp BTC/USD historical aggregate, representing one of the earliest and most continuous exchange records. Data points prior to Day 1242 rely on earlier, fragmented market aggregates (e.g., Mt. Gox or historical synthetics) and inherently contain a slightly higher degree of statistical noise due to early-stage liquidity constraints.

6.4 Temporal Translation (Phase Shift) and Cyclical Nodes

As theoretically outlined in Section 4, applying a constant temporal offset to the baseline geometry allows us to test the structural validity of the model against the asset's macro-cycle tops. Using a phase shift of $\Delta t = 2000$ days, the identically curved geometry interacts with three major cyclical extremes.

While isolating only a small number of contact points does not constitute absolute mathematical proof of a universal rule, their precise geometric alignment with the unmodified curvature of the baseline function shifts the balance of probability away from mere coincidence.

Days (x)	Actual Top (USD)	Shifted Model $f(x + 2000)$	Log Residual ($\Delta \ln$)
3270	19666	17420	-0.121
4452	61782	61525	-0.004
4484	64895	63513	-0.022

*** Note on Phase Shift Accuracy:** Applying the same Mean Absolute Centered Error (MACE) methodology to these three cyclical tops yields a shape variance of approximately 0.048 (4.8%). While a three-point sample size is inherently too small to draw definitive statistical conclusions, this minimal geometric error further demonstrates the structural alignment of the tangent function even when translated blindly across multi-year market extremes.

6.5 Independent Verification and Data Visualization

For independent verification against live market data, readers can apply the following Pine Script to overlay the baseline and temporal shift functions directly onto a logarithmic Bitcoin chart on TradingView (simply add the script to your chart as an indicator, and it will overlay directly on the price data):

```
//@version=6
indicator(title="Tangent-Based Model & Power Law Models", shorttitle="BTCtgX", overlay=true)

A = input.float(5.8535e-17, title="A", group="Tangent-Based Model Parameters")
B = input.float(7.4893e-5, title="B", group="Tangent-Based Model Parameters")
n = input.float(5.4666, title="n", group="Tangent-Based Model Parameters")

APwL = input.float(5.8535e-17, title="A", group="Power Law Model Parameters (Equivalent)")
nPwL = input.float(5.4666, title="n", group="Power Law Model Parameters (Equivalent)")

APL = input.float(2.0616e-17, title="A", group="Optimized Power Law Parameters (OLS)")
nPL = input.float(5.6218, title="n", group="Optimized Power Law Parameters (OLS)")

showHigh = input.bool(true, title="Show Shifted Curve (High)", group="Display Settings")
showLow = input.bool(true, title="Show Base Curve (Low)", group="Display Settings")
showPowerLaw = input.bool(true, title="Show Equivalent Power Law Curve", group="Display Settings")
showOptPowerLaw = input.bool(true, title="Show Optimized Power Law Curve", group="Display Settings")
```

```

timeShift = input.int(2000, title="Time Shift (Days)", minval=0, group="Time Settings")

genesis_time = timestamp(2009, 1, 3, 0, 0, 0)

daysElapsed = (time - genesis_time) / 86400000

var float modelPrice = na
var float modelPriceShift = na
var float powerLawPrice = na
var float optPowerLawPrice = na

if daysElapsed > 0
    angle = B * daysElapsed
    baseVal = math.tan(angle) / B
    modelPrice := A * math.pow(baseVal, n)

    daysElapsedShift = daysElapsed + timeShift
    angleShift = B * daysElapsedShift
    baseValShift = math.tan(angleShift) / B
    modelPriceShift := A * math.pow(baseValShift, n)

    powerLawPrice := APwL * math.pow(daysElapsed, nPwL)

    optPowerLawPrice := APL * math.pow(daysElapsed, nPL)

isBTC = syminfo.ticker == 'BTCUSD' or syminfo.ticker == 'BTCUSDC' or syminfo.ticker == 'BTCUSDT'

plot(isBTC and showLow ? modelPrice : na, color=color.green, linewidth=1, title="Baseline (Tangent)")
plot(isBTC and showHigh ? modelPriceShift : na, color=color.red, linewidth=1, title="Shifted (Tangent)")
plot(isBTC and showPowerLaw ? powerLawPrice : na, color=color.gray, linewidth=1, title="Power Law (Equivalent)")
plot(isBTC and showOptPowerLaw ? optPowerLawPrice : na, color=color.purple, linewidth=1, title="Power Law (Optimized)")

```

Pine Script code for Figure 1.



Figure 1.

Green: Tangent-Based Curve (Baseline)
Red: Tangent-Based Curve (Shifted)
Gray (in the Pine Script): Equivalent Power Law Curve
Purple (in the Pine Script): Optimized Power Law Curve

7. LIMITATIONS

A frequent objection to frameworks of this kind is that they are difficult to falsify. For a single, fixed asymptotic curve, however, this concern is somewhat overstated: **there is no indefinite middle ground between validation and falsification**. If a given calibrated curve is genuinely asymptotic, continued observation will, sooner or later, either keep confirming that curve or decisively break it — a fixed asymptotic curve cannot remain ambiguous indefinitely.

Mathematically, this lack of indefinite ambiguity can be formalized by evaluating the absolute divergence between the asymptotic model $f(x)$ and a continuous, non-asymptotic alternative, such as the standard Power Law $P(x)$. If $f(x)$ possesses a vertical asymptote at a finite time x^* , while $P(x)$ remains finite, the distance between them mathematically must diverge to infinity:

$$\lim_{x \rightarrow x^*} |f(x) - P(x)| = \infty$$

By the formal definition of a limit, this guarantees that for any arbitrarily large acceptable error margin E , there exists a critical time threshold $x_E < x^*$ beyond which the divergence strictly exceeds E :

$$|f(x) - P(x)| > E \quad \text{for all } x > x_E$$

In simpler terms: the geometric separation between the finite Power Law curve and the singular tangent curve accelerates so violently as $x \rightarrow x^*$ that they cannot mathematically remain within a plausible margin of error of one another. The passage of time forcefully breaks the statistical overlap, forcing a definitive resolution.

It is important to be clear about what this eventual separation does, and does not, establish. It guarantees only that these two specific curves cannot remain indistinguishable forever — it is a statement about the long-run behavior of two fixed mathematical objects, not a validation or falsification test for this paper's parameterized model. **Continued tracking of this paper's specific curve, even if it happened to hold precisely, would not by itself confirm that curve's validity**, since the space of curves capable of producing such tracking is considerably larger, and easier to satisfy after the fact, than a single confirmed prediction would suggest. The remainder of this section develops this distinction — between *this paper's specific parameterization* and the *general family of curves* it instantiates — in stages: an illustration of parameter sensitivity (7.1), the relativity of disruption across nested time-horizons (7.2), an extreme disruption viewed from the perspective of the curve as a measurement structure (7.3), the domain within which the framework applies (7.4), and the identifiability of the latent exponent (7.5). Section 7.6 then turns to a simulation direction in which the exponent itself fluctuates.

7.1 A Worked Example of Parameter Instability

To illustrate this sensitivity with a concrete case: an independent refitting of this model using seven refined historical price points (rather than the original six) produced a materially different parameter set — $A = 5.8535 \times 10^{-17}$, $B = 7.4893 \times 10^{-5}$, $n = 5.4666$, compared to the originally published $A = 10^{-16}$, $B = 8.3 \times 10^{-5}$, $n = 5.395$. Despite B alone shifting by roughly 10%, the implied singularity date moved from Day 18,925 (~2060) to Day 20,974 (~2066) — a shift of several years arising from what appears, at the level of individual parameters, to be a minor refinement.

This is a direct empirical demonstration of a broader point: **the model's fit to historical nodes is comparatively stable** under reparameterization — both parameter sets track the seven calibration nodes closely — **while the extrapolated asymptote is considerably more sensitive**. The geometric shape of the curve across the observed historical range remains broadly consistent between refits; the precise forward-projected date does not share that stability. **Consequently, the specific parameters presented in this paper (A, B, n) should not be treated as fixed, immutable constants** — they are the current best descriptive calibration, subject to revision as further data emerges. The same caution applies to the reflexivity exponent itself: **the choice of y^2 in Section 3.3 is a modeling simplification, not a fixed physical constant**, and is examined further in Section 7.5.

7.2 Nested Time-Prefixes, Black Swans and Antifragility

Every observation takes place within a given **time-prefix**: a window whose length determines which events appear ordinary and which appear extreme. An event that is rare, has a large impact, and could not have been anticipated from within that window is what Taleb calls a **Black Swan** (Taleb, *The Black Swan*, 2007). As Taleb emphasizes, this designation depends on the observer's position and knowledge. **The same event can be a complete surprise within one frame and an expected occurrence within another.**

A fractal-like structure. The framework of this paper assumes that time-prefixes are nested inside one another, and that similar dynamics recur across these scales. This view is closely aligned with Benoit Mandelbrot's work on self-similarity and fractal geometry (Mandelbrot, *The Fractal Geometry of Nature*, 1982). Studying river flows and precipitation records, Mandelbrot and Wallis described two properties of such systems: extreme events can be very extreme (the "Noah Effect"), and runs of unusually high or low values can be very long (the "Joseph Effect") (Mandelbrot & Wallis, *Water Resour. Res.* 4, 909, 1968). The simulation of Section 7.6 shows a qualitative counterpart to the second property: **the more persistent the fluctuations, the sooner the finite-time singularity arrived**. The persistence of its noise is simple and finite, however, and does not reproduce the long-range dependence of Mandelbrot's self-similar models.

Nested functions as a possible source of noise. In this fractal-like picture (Part II, Section 4.3, Simulation), the noise of a slower base function would be supplied, at least in part, by the smaller, faster functions nested inside it (Section 7.6.6). Each nested function accelerates, reaches its own asymptote and ceases to exist, while the observed level returns to the base beneath it. Seen from within the short time-prefix of a nested function, its end is an extreme, terminal event. Seen from the time-prefix of the base function, the same event is only one of many excursions that together

make up its noise. **Events of this kind, which do not break the larger time-prefix, strengthen its structure rather than weaken it.** In the construction, each excursion leaves the underlying state permanently higher, and the accumulated excursions drive the base function toward its own asymptote (Sections 7.6.5 and 7.6.6).

Consistency with Taleb's framework. This is antifragility in Taleb's sense: a system that gains from variability, shocks and disorder, rather than merely withstanding them (Taleb, *Antifragile*, 2012). Two further elements of that framework are preserved here. First, Taleb notes that the antifragility of a larger system often rests on the fragility of its smaller parts. In the nested picture, it would be precisely the termination of the nested functions that supplies the gains of the base function. Second, antifragility holds only up to a certain intensity of shocks. **The construction therefore does not eliminate Black Swans; it locates them relative to scale.** An event large enough to end the base function itself would be a Black Swan at that level as well, and beyond that point the structure it supported would cease to exist (Section 7.6.7). Nor does the construction make such events predictable: even in the simulation, the timing of the singularity varied widely from run to run.

Scope. This section describes a conceptual structure. The nesting has not been simulated explicitly across multiple levels (Section 7.6.8); the simulation replaces the nested functions with a simplified noise model. The parallels above are offered as structural correspondences, not as evidence that any particular real-world system is organized in this way.

7.3 The Parameterized Curve as a Measurement Structure: What an Extreme Event Does to It

The parameterized curve of Section 6 is best treated as **a measurement structure**. It is calibrated on a set of tangency nodes within a specific historical stretch, between Day 745 and Day 6388, and its reading, the effective feedback exponent of Section 3.3.1, refers to that stretch. Like any calibrated instrument, it has a **calibration range**, and its reading is only as well determined as its calibration points allow. An extreme scenario makes this concrete, using the tangency rules of Section 6.1: suppose price collapsed to \$0.001 at day 7,000.

Applying the non-crossing and tangency conditions strictly, this scenario produces a deterministic, not arbitrarily chosen, outcome. Testing shows that no coherent tangent curve can simultaneously satisfy exact tangency at all seven previously established historical nodes *and* this new point; any attempt produces errors of several hundred percent and a degenerate, non-physical implied singularity. **Under the model's own rules, this forces the exclusion of the prior nodes as valid tangency points once this new data exists.** What survives as valid anchoring points is only the Genesis origin and this single new node. This holds not only against the seven curated nodes discussed in this paper, but, in principle, against the full historical daily price record, which a disruption of this magnitude would similarly fail to remain consistent with.

The current calibration would not survive such an event. The published parameters ($A = 5.8535 \times 10^{-17}$, $B = 7.4893 \times 10^{-5}$, $n = 5.4666$), and the effective exponent derived from them, would remain a description of the stretch on which they were calibrated. They could not, however, be carried across the event. In measurement terms, the event lies outside the calibration range.

After the event, the instrument has no determined reading. With only the Genesis origin and one new point constraining the system, the exponent n and the curvature parameter B are both free. Because $f(0) = A(\tan(0)/B)^n = 0$ automatically, for *any* choice of A , B and n , the Genesis condition imposes no real constraint at all, and the entire remaining freedom is available to satisfy the single new data point. Given any desired future singularity day x^* and any chosen exponent n , the required parameters can be solved directly:

$$B = \frac{\pi}{2x^*} \quad A = \frac{y_{target}}{(\tan(B \cdot x_{target})/B)^n}$$

For any reader wishing to verify this independently: substituting $x_{target} = 7000$, $y_{target} = 0.001$, and any exponent n (for example, 3, 5, or 10) into this formula, for any chosen future singularity day x^* beyond day 7,000, always yields a valid parameter pair reproducing the required point exactly, with f at Genesis equal to zero by construction. This freedom runs in *both* directions: x^* can be pushed later, but it can equally be pulled earlier than the current calibration implies. Since n is free as well, **the effective feedback exponent is left entirely undetermined.**

The conclusion is therefore a statement about measurement, not about resilience. An extreme disruption ends the validity of the current calibration and leaves the instrument without a determined reading, because a single new node cannot fix both B and n . A new measurement of the effective feedback exponent becomes possible only once enough new tangency nodes have formed after the event to constrain the parameters again. That measurement would then describe the new stretch, not the old one. **A difference between the two readings would itself be informative:** it would indicate that the effective feedback changed across the event. This is consistent with Section 2.1, where fitted constants describe an observed trajectory rather than fixed constants of the process. Conversely, the freedom shown above means that curve-fitting alone cannot exclude the qualitative structure. **This is a limitation of the instrument, not evidence in favor of the structure** (Sections 7.4 and 7.5).

7.4 Domain of the Model: The Boundary at Zero

The governing equation introduced in Section 3.3, $dy/dx = k(1+y^m)$, has $1+y^m > 0$ for any non-negative y and any m , with $k > 0$. This means dy/dx is strictly positive everywhere in the model's domain, for every value of m . **On its own, the equation is structurally incapable of producing a decrease, let alone a decline toward zero.** Consequently, $f(x) > 0$ holds by construction at every point after Genesis. A price of exactly zero is

therefore **outside the domain of this model**. The model has nothing to say about it, in the same sense that it has nothing to say about a discrete external event such as an exchange delisting or a network shutdown. For the same reason, a price of zero is not a falsification test in either direction: a genuine test requires a condition the model can meaningfully fail, and a monotonically increasing function offers none in this direction.

Within this domain, the model's reach is wide. As shown in Section 7.3, the Genesis condition $f(0) = 0$ is satisfied automatically for any choice of parameters, so the remaining freedom (A , B , and the exponent, whether fixed at n or generalized to any $m > 1$ in the underlying term) can always match a single new touch point exactly. This does not depend on the price level of that point, and it applies again to every subsequent touch point, however often a collapse and recovery repeats. **No observed price path that stays strictly above zero can therefore exclude every member of this functional family.** This is a statement about the extent of the domain, not support for the model. It also means that price data alone cannot rule the family out, nor establish that $m > 1$ (Section 7.5).

This picture applies to a single function, and it becomes more nuanced for nested functions. In the nested construction of Sections 7.6.6 and 7.6.6.1, the underlying state still never decreases. The observed moving average, however, can fall back onto the function beneath it, and once the outermost base function reaches its own asymptote, it can fall to zero. In that construction, zero is reached not through a gradual decline of the state, but through the end of the structure that supported it.

7.5 The Latent Exponent Is Likely Unidentifiable from Price Data Alone

A further limitation concerns the choice of $m=2$ itself in the latent equation $dy/dx=k(1+y^m)$. Repeating the derivation of Section 3.3.1 for general m shows that, near the singularity, the effective feedback exponent in price terms is $E = 1 + (m-1)/n$ — which reduces to $1+1/n$ only at $m=2$. Any other $m > 1$ paired with a correspondingly rescaled $n = (m-1)/(E-1)$ reproduces the *same* effective exponent, and therefore plausibly the same near-singularity behavior.

This offers a limited defense of the model: the specific value $m=2$ is likely not a substantive, falsifiable commitment, since the empirically observable quantity is the effective exponent, not m directly — choosing $m=2$ for closed-form convenience does not obviously cost anything in fit quality. **But this cuts both ways:** it also means m is likely unidentifiable from price data alone, since any $m > 1$ can plausibly be rescaled via n to match the same observed curve. This equivalence has only been established asymptotically, near the singularity; whether it extends to the full historical trajectory, including the early, near-power-law regime, has not been tested and remains an open question.

7.6 A Possible Simulation Direction: The Exponent as a Stochastic, Regime-Switching Process

This subsection documents, in purely mathematical terms, one possible direction for future simulation work: a generalization of the reflexivity equation in Section 3.3 to a time-varying, stochastically-driven feedback exponent, rather than the fixed value used in the paper's baseline model. No claim of any kind is made here about what the underlying random variable represents, or that this construction describes Bitcoin specifically; the signal is treated strictly as an abstract binary variable, offered as a starting point for further exploration.

7.6.1 The driving signal

Let $S = (S_1, S_2, \dots, S_N)$ be a sequence of independent, identically distributed discrete random variables, each taking one of two values:

$$S_i = \begin{cases} 2 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

for $i = 1, \dots, N$, with a single asymmetry parameter $p \in (0,1)$. By construction:

$$E[S_i] = 2p \quad \text{Var}[S_i] = 4p(1-p)$$

A single driving signal is sufficient. Only the feedback exponent m enters the differential equation of Section 7.6.3; the mapping exponent n does not. The construction therefore uses one noisy signal, which drives m , and holds n fixed at 1.

7.6.2 The moving average and the driving exponent

Fix a window length W , a positive integer. Define the centered moving average of the sequence S with window W as:

$$M_W[S](i) = \frac{1}{W} \sum_{|j-i| \leq W/2} S_j$$

The two exponents of the model are then set as:

$$m(i) = M_W[S](i) \quad (\text{stochastic, slow-varying})$$

$$n \equiv 1 \quad (\text{deterministic constant})$$

By the law of large numbers, as $W \rightarrow \infty$, $M_W[S](i) \rightarrow 2p$ almost surely. The exponent $m(i)$ therefore fluctuates around the long-run mean $2p$, with $\text{Var}[M_W[S]] \approx 4p(1-p)/W$ in the i.i.d. case; with the holding times introduced in Section 7.6.4, $\text{Var}[M_W[S]] \approx 4p(1-p) \cdot 2H/W$. **The amplitude of the fluctuations is thus governed by the ratio H/W** , that is, by the persistence of the generator relative to the length of the window, and not by either quantity on its own.

7.6.3 The governing equation

Let $y(x)$ solve the non-autonomous ordinary differential equation with piecewise-constant, stochastic coefficient $m(x)$:

$$\frac{dy}{dx} = k(1 + y^{m(x)}), \quad y(0) = y_0 > 0 \text{ small}, k > 0 \text{ constant}$$

and define the observable quantity:

$$f(x) = A \frac{y(x)}{B} \quad (n = 1)$$

for constants $A, B > 0$. **The mapping exponent is held fixed at $n = 1$** , so the observable is a linear rescaling of the state. This is not merely a simplification of convenience: n does not appear in the differential equation at all, and a monotone rescaling of y cannot move the point at which y diverges. **The location of the singularity x^* is therefore determined by $m(x)$ and the rate k , and not by the mapping**; since k is held fixed, every stochastic result reported below originates in that single noisy exponent. Since $m(x)$ is piecewise constant (constant over each unit interval of x , by construction of the moving average on integer-indexed data), the equation for y is solved by numerically integrating forward across each interval with the locally-constant exponent, then updating m at the next interval; this requires no assumption beyond local (sub-interval) numerical stability of the chosen integration scheme.

7.6.4 Observed Result: A Noise-Driven Finite-Time Singularity

The construction in Sections 7.6.1–7.6.3 was simulated with one generalization of the driving signal, anticipated in Section 7.6.2: **the generator holds its current state for a random number of steps before redrawing**, rather than drawing a new, independent value at every step. Holding times are geometrically distributed with mean H . At each switching epoch a new value is drawn with the same probabilities as in Section 7.6.1 (2 with probability p , 0 otherwise). The long-run mean $E[S_i] = 2p$ is therefore unchanged. What changes is how long the moving average can remain on one side of the critical boundary $E[S] = 1$. The i.i.d. construction of Section 7.6.1 is recovered as the special case $H = 1$.

The central test tilts the generator slightly *below* the critical value:

Quantity	Driving signal $m(x)$
Asymmetry parameter p	0.48 (long-run mean $0.96 < 1$)
Window length W (days)	1461
Mean holding time H (days)	40
Standard deviation of the moving average	≈ 0.23
Mapping exponent n	fixed at 1, noise-free

Remaining constants: $k = 10B \approx 7.4893 \times 10^{-4}$, $y_0 = 10^{-6}$, with A and B taken from Section 6.2. With this value of k , the reference configuration (generator tilted upward to $p = 0.6$) produces a finite-time singularity at a median of approximately day 5100, the same order as the singularity reported in earlier runs of this system (day 5778). The equation for y was integrated in logarithmic form, $u = \ln y$. Once the feedback term dominated the constant term, each unit interval was solved in closed form with its locally constant exponent, including the exact location of the blow-up within that interval. All results below are sample statistics over independent random seeds.

Configuration (p_m)	Runs	Runs reaching a singularity	Median x^* (days)	5th–95th percentile of x^*
Reference: 0.60	200	100%	≈ 5100	—
Sub-critical: 0.48	200	100%	$\approx 11\,290$	$\approx 5390 - 22\,260$

Every run of the sub-critical configuration ended in a finite-time singularity, even though the exponent's long-run mean (0.96) lies below the critical boundary. The downward tilt did not remove the singularity. It only delayed it, roughly doubling the median time to blow-up relative to the upward-tilted reference.

The role of noise can be isolated directly. If the fluctuations are removed and the exponent is held at its mean, $m(x) \equiv 2p = 0.96$, the equation has no finite-time singularity. For large x it then grows only as a power law:

$$y(x) \approx [(1 - m) \cdot k \cdot x]^{1/(1-m)}, \quad \text{finite for every finite } x$$

The deterministic, noise-free system is therefore stable, while the same system with the same mean exponent but with fluctuations is not. **The singularity is produced by the fluctuations themselves, not by the average tilt.** The following table varies only the speed of the generator, and hence the amplitude and persistence of the fluctuations in the moving average. Each cell gives the median day of the singularity over 30 runs, over a horizon of 1 000 000 days, and every run in every cell reached a singularity.

Mean holding time H (days)	Std. dev. of $m(x)$	$p_m = 0.48$	$p_m = 0.45$	$p_m = 0.40$
10	≈ 0.11	26 591	48 992	171 267
20	≈ 0.16	17 777	25 521	71 959
40	≈ 0.23	11 428	14 608	24 134
80	≈ 0.32	7 952	9 254	12 183

Two regularities are visible:

- For a fixed tilt, a slower generator produces larger and more persistent fluctuations, and the singularity arrives earlier.
- For a fixed generator speed, a stronger downward tilt delays the singularity but, across the tested range down to $p = 0.40$, does not prevent it.

The i.i.d. construction ($H = 1$) is the low-noise limit of the same system. There the moving-average fluctuations are an order of magnitude smaller (std. dev. of $m(x) \approx 0.03$), and at $p_m = 0.48$ only 7 of 20 runs reached a singularity within 400 000 days. Over the full horizon all 20 eventually did, but very late (median ≈ 448 000 days). The ordering is therefore monotone:

no noise (no singularity) → weak, fast noise (rare and very late singularity) → strong, slow noise (early singularity)

This replaces the earlier reading of the parameter $p^* = 1/2$ as a sharp boundary between a singular and a non-singular regime. In the presence of noise, p^* does not decide *whether* a singularity occurs. It shifts *when* it occurs, together with the amplitude and persistence of the fluctuations. Below p^* , the power-law-like growth described above persists only as a transient phase, whose length depends on how rarely the moving average makes a sufficiently long excursion above the critical boundary.

7.6.5 Mechanism: Why Fluctuations Force the Singularity, and the Link to Antifragility

The mechanism follows directly from the governing equation. During any interval in which the exponent stays constant at a value $m > 1$, the time the trajectory needs to reach infinity, starting from its current level y , is at most:

$$\Delta x^* \leq \frac{y^{1-m}}{k(m - 1)}$$

(the $+1$ term in the equation can only shorten this time). Because $1 - m < 0$, this required duration **shrinks toward zero as y grows**. At the same time, as shown in Section 7.6.7, $dy/dx \geq k > 0$ at every point, so y never decreases, whatever regime the exponent is in. The consequences follow in sequence:

- Periods with $m(x) < 1$ slow the growth but never reverse it.
- Periods with $m(x) > 1$ that are too short to cause a blow-up still lift y to a permanently higher level.
- From that higher level, an even shorter excursion above the boundary suffices next time.

The trajectory therefore climbs in a stepwise fashion: long, nearly flat stretches are interrupted by sudden jumps coinciding with excursions of $m(x)$ above 1. Eventually one excursion is long enough relative to the current level, and the system reaches its finite-time singularity. As long as the moving average has a nonzero probability of staying above 1 for a sustained interval, which holds for any finite window length, this outcome is only a matter of time. This is why the noise-free system (Section 7.6.4) is the only fully stable case.

The same conclusion can be stated as a convexity property. At a fixed level $y > 1$, the growth term $y^m = e^{m \cdot \ln y}$ is a convex function of the exponent:

$$\frac{\partial^2}{\partial m^2} y^m = (\ln y)^2 \cdot y^m > 0$$

Upward fluctuations of m therefore add more to the growth rate than equally large downward fluctuations take away (Jensen's inequality). The time-averaged growth rate of the fluctuating system consequently exceeds that of the noise-free system with the same mean exponent. **Because the curvature grows with $(\ln y)^2$, this asymmetry strengthens as the system itself grows:** the larger the system becomes, the more it gains from each fluctuation.

This is precisely the formal signature of antifragility in Taleb's sense: a convex response to variability, under which the system benefits from volatility, disorder, and shocks rather than merely withstanding them. The simulation thus provides a quantitative counterpart to the conceptual description of antifragility given in Section 7.2. **In this system, noise, that is, shocks, is not an obstacle to the upward dynamic but its driver.** Two features make this explicit:

- The same mean exponent produces a stable, power-law path without noise, yet a finite-time singularity with it.
- Stronger and more persistent fluctuations bring the singularity earlier.

At the same time, the zero boundary of Section 7.6.7 guarantees that downward fluctuations can never push the underlying state back. The system is thus exposed asymmetrically: it keeps the gains from its upward excursions and cannot be eroded by its downward ones. As with the rest of Section 7.6, this is a statement about the mathematical construction, not a claim that the abstract signals represent any specific real-world quantity.

The speed of the generator matters as much as its tilt. The results in Section 7.6.4 show that the holding time, meaning how long the generator remains in a given state, is at least as important as the average asymmetry in determining when the singularity occurs. In a nested, multi-level treatment, the holding time would plausibly grow longer at the slower, outer levels of the hierarchy. By the mechanism above, this would make the slowest levels the most consequential ones, since their excursions are the most persistent. **A qualification from the second part of this work:** in this construction, a slower generator also enlarges the fluctuations of the moving average (Section 7.6.2), so persistence and amplitude change together. When the two are separated, amplitude appears to be the dominant factor, and persistence matters mainly by making sufficiently long excursions possible at all (Part II, Section 3).

The present construction contains a single fluctuating level, and the asymmetry of its generator was fixed exogenously. In a fuller treatment, the asymmetry parameter and holding time of each level would plausibly be weighted functions of the states of the other levels, with every level, at every timescale, exerting some influence on every other. Specifying these weights is left for further work; a conceptual construction is outlined in Section 7.6.6.

7.6.6 Nesting of Functions: Supporting Structures as a Possible Source of Noise

The random generator of Section 7.6.1 is a simplification. This section describes, at a conceptual level, one possible reading of what it stands in for: **a hierarchy of nested functions of the same qualitative kind, in which the smaller, faster structures would supply the fluctuations of the larger, slower ones. Where the fluctuations actually come from is not settled here.** What the simulations show at the level of a single function is a high-frequency, structureless fluctuation whose source cannot be localized, and which behaves, as far as the equation is concerned, much like a random generator. **Terminating nested functions are a plausible contributor to it, not an established one,** and none of the results of Section 7.6.4 depend on the answer. **The question is taken up in a form that can be examined rather than only described in Part II, Section 4.3,** whose simulation carries both an external fluctuation and a coupling by which the completion of a faster instance acts on the exponent of the one beneath it. The two can be enabled separately, so that the contribution of each is visible; **what is offered conceptually here is there a parameter.**

Construction. Consider a large, long-horizon base function, such as the function $f(x)$ of this paper, calibrated on the tangency nodes of Section 6. Select a point x_0 on this function and treat it as a new origin. From this origin a second function g starts, with local time $\xi = x - x_0$. This function has the same superlinear, asymptotic geometry, but a much faster rise: its own asymptote lies at a local time ξ_g^* far shorter than the remaining distance to the base function's asymptote,

$$\xi_g^* \ll x^* - x_0$$

A faster moving average is taken over this structure. Let $L(x)$ denote the level of the innermost function that exists at x : $L(x) = g(x - x_0)$ while g is defined, and $L(x) = f(x)$ otherwise. The moving average is taken on a logarithmic scale, over a window W_M that is short compared with the lifetime of the base function:

$$\ln M(x) = \frac{1}{W_M} \int_{x-W_M}^x \ln L(s) ds$$

The moving average follows g upward, but **it does not reach the asymptote of g** . The logarithm of a function diverging at a finite point remains integrable, so $M(x)$ stays finite as g diverges. The moving average therefore detaches from g as g approaches its asymptote. Beyond the asymptote, at $\xi > \xi_g^*$, the smaller function no longer exists. The window of the moving average fills up with values of the function beneath, and $M(x)$ falls back onto the base function $f(x)$. The base function therefore acts as a **supporting structure** for the moving average: the nested excursions rise above it and return to it, but do not break through it. **This support lasts only as long as the base function itself exists.** The base function has the same asymptotic geometry as the structures nested in it, so it also terminates at its own asymptote x^* . What happens once the outermost base function itself ceases to exist is taken up in Section 7.6.7. Writing each such excursion as a non-negative log-deviation D_j from the base, starting at its own origin x_j , the moving average can be written schematically as:

$$\ln M(x) \approx \ln f(x) + \sum_j D_j(x - x_j), \quad D_j \geq 0$$

Here D_j is the windowed average of $\ln(g_j/f)$ over the part of the window in which g_j exists. The approximation reflects that $\ln f$ itself changes slowly across the short window. The profile D_j is zero before its origin. It then rises along the accelerating path of its own function g_j , detaches as g_j reaches its asymptote, and decays back to zero once g_j has ceased to exist.

The construction is recursive: each g_j can itself serve as the base function for still smaller and faster nested structures, which in turn would supply its fluctuations. Read this way, **the sum of the nested excursions below a given level would be that level's noise**, and the generator $m(x)$ of Section 7.6.1 a statistical stand-in for that aggregate. Its holding time (Section 7.6.4) corresponds to how long a nested structure keeps accelerating before the moving average detaches from it.

The validity of the parameters comes to the fore again. In this picture, the parameters of the base function (Section 6.2) are not fixed constants of a single process. They are effective averages over everything nested beneath it. The parameters of the nested structures themselves (origin, speed, lifetime, amplitude) are unstable and vary from one excursion to the next. This is the same fragility of parameters emphasized throughout Section 7, now appearing as a structural feature rather than as a limitation of the fit.

Instability of the parameters alone is sufficient for an asymptote. The single-level simulation of Section 7.6.4 supports this directly. Setting the generator exactly to the critical value, $p_m = 0.5$, gives the mean exponent $E[m] = 1$. All other settings were as in Section 7.6.4.

Exponent with mean $m = 1$	Runs	Runs reaching a singularity	Median x^* (days)	5th–95th percentile of x^*
No noise ($m \equiv 1$)	—	none (pure exponential growth)	—	—
Fast, weak noise (i.i.d., $H = 1$)	20	100%	$\approx 70\,420$	—
Slower, stronger noise ($H = 40$)	200	100%	$\approx 8\,850$	$\approx 4\,120 - 18\,910$

Without fluctuations, the system with mean exponent 1 has no finite-time singularity: the equation reduces to $dy/dx = k(1 + y)$, whose solution grows exponentially and remains finite for every finite x . With a fluctuating exponent of exactly the same mean, every run reached an asymptote, and the more persistent the fluctuations, the sooner it arrived. **A function whose parameters are unstable, but whose average feedback exponent is exactly $m = 1$, can therefore still terminate in an asymptote.** In the nested picture, the fluctuating structures beneath it are what make this happen.

Detachment and accumulation. In the nested picture, the moving average never completes the asymptotes of the nested levels: each nested function reaches its own asymptote and ceases to exist, while the moving average detaches and falls back onto the level beneath (Section 7.6.6.1). In the terms of Section 7.6.5, this is consistent with how the asymptote of the level that contains them is nevertheless reached. Excursions of the exponent above the critical boundary that are too short to cause a blow-up do not vanish without trace: they leave the underlying state y permanently higher. The moving average returns to its supporting base, but the excursion's contribution to the underlying state is not erased (Section 7.6.7). Asymptotes that the moving average does not complete at the nested levels thus accumulate into the completed asymptote of the level that contains them. This is a further expression of the antifragile structure described in Section 7.6.5: on this reading, the shocks arriving from the nested levels are what drive the larger structure forward.

As stated in Section 7.6.8, this nesting has not been simulated explicitly across multiple levels here. The random generator remains a simplified substitute for it. The construction above is offered as the interpretation that the generator is meant to approximate, and as a concrete starting point for a fully nested simulation.

7.6.6.1 A simple numerical illustration

The following script, run in JupyterLab on a Fedora Workstation system using only NumPy, pandas and Matplotlib, reproduces the two effects described in this section on a single level. It is not the simulation of Section 7.6.4: **the parameters differ, and the exponent here is drawn independently on each step rather than taken from a moving average of a driving signal**, so the fluctuations carry no persistence at all. The exponent alternates between 0.8 and 1.2, with the higher value drawn with probability 0.45, **giving a mean exponent of 0.98, below the critical value of 1**. The state nonetheless reaches a finite-time singularity, which illustrates the central point of Section 7.6: the outcome is decided by the excursions above the boundary, not by the average. It also shows why persistence is not indispensable here. As the state grows, the superlinear stretch required to complete the divergence becomes shorter, until even the brief runs produced by independent draws are long enough. The second half of the script forms the observed quantity and its logarithmic moving average, and **plots the moment at which the function ceases to exist and the moving average detaches from it**, falling back toward the base level drawn beneath it.

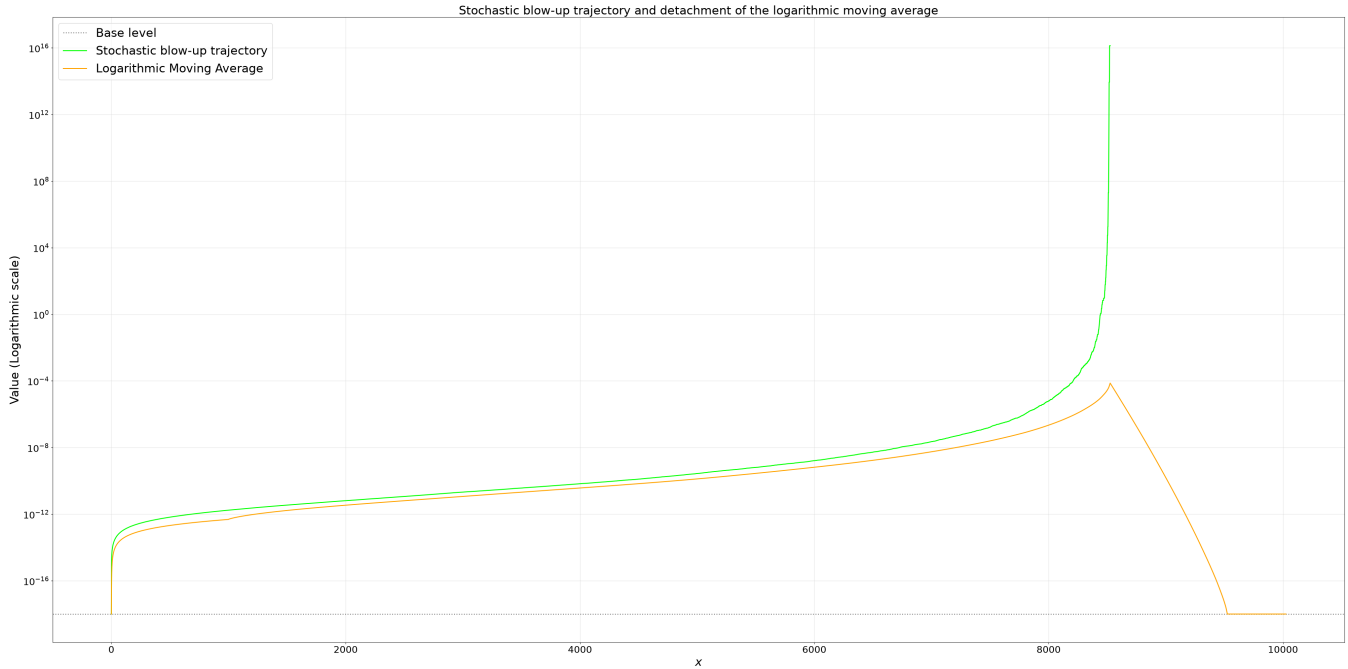


Figure 2.

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

# --- 1. PARAMETERS ---
A = 10e-17
B = 10e-5
k = 10 * B
y0 = 1e-6
windowM = 1000      # Moving average window size in days

# Noise configuration
p = 0.45
numDays = 20000

# --- 2. NOISE GENERATION ---
m = np.random.choice([1.2, 0.8], size=numDays, p=[p, 1-p])

# --- 3. SOLVING THE DIFFERENTIAL EQUATION ---
y = np.zeros(numDays)
y[0] = y0

blowupDay = numDays # Store the day when the blow-up occurs

for i in range(1, numDays):
    dy = k * (1 + y[i-1]**m[i-1])
    y[i] = y[i-1] + dy

    # Safety break: singularity (blow-up)
    if y[i] > 1e30:
        blowupDay = i
        # The latent state drops back to the base level (y0) to compute the moving average decay.
        y[i:] = y0
        print(f"Singularity (blow-up) reached on day {i}! The function ceased to exist.")
        break

# --- 4. OBSERVABLE QUANTITY AND MOVING AVERAGE ---
```

```

f = A * (y / B)
baseLevel = A * (y0 / B) # The starting height (supporting structure)

# Calculating the logarithmic moving average on the latent state
logF = np.log(f)
logM = pd.Series(logF).rolling(window=windowM, min_periods=1).mean()
M = np.exp(logM)

# For plotting, the original function genuinely disappears after the singularity
fVisible = f.copy()
fVisible[blowupDay:] = np.nan

# --- 5. PLOTTING ---
lastVisibleDay = min(blowupDay + windowM + 500, numDays)

plt.figure(figsize=(40, 20))

# Plotting the horizontal base level
plt.axhline(baseLevel, color='gray', linestyle=':', linewidth=2, label='Base level')

# Plotting f(x) (which disappears) and the moving average
plt.plot(fVisible[:lastVisibleDay], color='lime', alpha=1, linewidth=2, label='Stochastic blow-up trajectory')
plt.plot(M[:lastVisibleDay], color='orange', alpha=1, linewidth=2, label='Logarithmic Moving Average')

plt.yscale('log')
plt.title('Stochastic blow-up trajectory and detachment of the logarithmic moving average', fontsize=25)
plt.xlabel('$x$', fontsize=25)
plt.ylabel('Value (Logarithmic scale)', fontsize=25)
plt.tick_params(axis='both', which='major', labelsize=20)
plt.legend(loc='upper left', fontsize=25)
plt.grid(True, which='both', alpha=0.3)

plt.tight_layout()
plt.show()

```

JupyterLab code for Figure 2.

Three points should be kept in mind when reading the output. First, the script shows the supported case, not the terminal one: after the singularity the function is removed from the plot, and the moving average decays over the length of its window toward the base level that stands beneath it. **The case in which no such level remains, and the moving average falls to zero, is the separate matter treated in Sections 7.6.7 and 7.4** Second, the integration is a fixed-step explicit scheme with a step of one day, so **the exact day of the singularity is scheme-dependent and only the qualitative behavior is meaningful**. Third, the random draws are not seeded, so individual runs differ; fixing a seed makes a particular run reproducible.

7.6.6.2 Implications for shorter horizons and for the parameterized model

This simulation suggests that, because of the instabilities described above, the approach of Sections 6 and 7.3, which reads the structure through a curve calibrated within a given time-prefix, is likely to be **considerably less accurate over shorter "trends" than over longer cycles**. At shorter horizons, the nested excursions and their unstable parameters make up a larger share of what is observed, while over longer cycles they tend to average out into the slower supporting structure. The simulation also raises a further possibility, stated here tentatively. The structural pattern examined earlier in this paper was observed through a model with **specific, fitted parameters**. The simulation indicates that a feedback exponent averaging $m = 1$ (or somewhat below 1; Section 2.1), with the mapping exponent held at $n = 1$ and combined with **a sufficient degree of noise**, may produce **structural properties similar to those of the parameterized model**, including termination in a finite-time asymptote. If this holds, the particular parameter values of Section 6.2 may be better read as one realization of a more general noise-driven structure than as its defining constants. This remains an interpretation of a deliberately simplified simulation (Section 7.6.8), not an established result.

7.6.7 A Note on the Zero Boundary

For a single function, this stochastic extension does not alter the zero-boundary property already established for the fixed-exponent case (Section 7.4): since $1 + y^{m(x)} \geq 1 > 0$ for every finite value $m(x)$ can take — including the (rare, but not impossible) case where the moving average of the driving signal, and with it the exponent, equals exactly zero — dy/dx remains strictly positive at every point along the trajectory, regardless of which regime $m(x)$ is currently in. **A single function therefore still cannot decrease on its own, let alone reach zero, under this generalization**, for the same structural reason as in the fixed-exponent case, not because of any special property of the moving-average construction itself. Within a single function, a price of zero remains outside the domain of the model.

This no longer holds for the nested system as a whole. In the nested construction of Section 7.6.6, the observed quantity is not the state of one function but the logarithmic moving average $M(x)$, which follows the innermost function that currently exists. When a nested function completes

its singularity, it ceases to exist, and the moving average detaches from it and falls back onto the supporting function beneath. **The observed level can therefore decline, even though no single function ever gives back any of its own growth.** Such declines are bounded from below only as long as a supporting function exists. The outermost base function has no support beneath it: **if it, too, ends in a singularity and the moving average detaches from it, the moving average falls to zero and the process as a whole stops.** Zero is therefore reachable in the nested system, but not through a gradual decline of any state variable; it is reached through the end of the last structure that supported the observed level. **The zero boundary of this section is a property of each function taken individually, not of the observed dynamics of the nested system.**

7.6.8 A Necessary Caveat on Simplification

While this simulation points in a direction that appears credible, it is only a very simplified model. It contains a single fluctuating level, a simplification made purely because of resource constraints, and it therefore does not handle the full set of nested timescales that a complete treatment of this idea would require.

7.6.9 Possible research or simulation directions

Possible directions include fractal geometry, biology, psychology, economic cycles, technological development, the hierarchical nesting of solar cycles, phantom dark energy cosmology (the "Big Rip" future singularity), and other systems exhibiting similar properties. A structurally related phenomenon appears in magnetic confinement fusion: in nonlinear ideal ballooning theory, the explosive onset of **tokamak edge instabilities** (edge localized modes), a key stability concern for next-generation devices such as ITER, is attributed to a finite-time singularity in the governing model (Wilson & Cowley, *Phys. Rev. Lett.* 92, 175006, 2004). There, a finite perturbation can trigger the transition even where the configuration is stable to infinitesimal ones. The parallel offered here is purely structural: superlinear feedback combined with fluctuations near a critical boundary. It is not a claim of shared physical mechanism.

This direction is noted at greater length than the others for a specific reason. Its connection to Bitcoin is indirect. It is included because the simulation of this section, together with the nested construction of Section 7.6.6, is not specific to any asset: it describes a general class of systems with superlinear feedback, fluctuating parameters and nested timescales. The discussion below is offered as an illustration of how far such a construction might reach, not as a contribution to plasma physics.

Where a downward path appears. The construction contains two levels that behave differently. The underlying state y has no downward path: by Section 7.6.7, $dy/dx > 0$ at every point. The observed level, however, does have one. In the nested picture of Section 7.6.6.1, the logarithmic moving average $M(x)$ follows a nested function g_j ; upward and detaches from it as g_j approaches its own asymptote. Once g_j has ceased to exist, **the moving average falls back onto the function beneath it.** Each excursion D_j therefore rises, peaks and decays, and the sequence of such excursions forms **a recurring build-up-and-crash cycle at the observed level.** The lower-level singularity is, in this sense, **regularized as seen from above:** the nested function diverges, but the moving average stays finite, because the logarithm of a function diverging at a finite point is integrable.

The lower bound of the moving average is zero, not the outermost function. The supporting role of each function holds only as long as that function itself exists. The outermost base function, the outermost level on which all others rest, is of the same qualitative kind as the ones nested within it: it also terminates in a finite-time asymptote. When it does, there is no further structure beneath it. As soon as the window of the moving average reaches beyond this point, **the moving average falls to zero: the process can stop.** This does not contradict Section 7.6.7. The underlying state never decreases; the path to zero does not lead through a decline of y , but through its completed singularity, beyond which the function no longer exists. The accumulation described in Section 7.6.6 therefore has a definite end point. Uncompleted asymptotes at the nested levels accumulate into the completed asymptote of the outermost level, and that completion is the termination of the whole structure.

Read this way, the construction contains **two kinds of downward movement.** The first is a partial crash, after which the moving average returns to a still-existing supporting function and the cycle continues. The second is a terminal crash, after which it falls to zero. The plasma case shows a structurally similar distinction. An edge localized mode only partially relaxes the edge profile, which then builds up again. A **disruption**, the abrupt loss of confinement that terminates the discharge, corresponds more closely to the terminal case.

The differences are equally important. In the plasma, the regularization of the partial crashes is a physical process: saturation, relaxation, and the loss of energy and particles from the edge. In the construction, it is a property of the averaging operator. The nested function does complete its asymptote; only the observed average fails to follow it. In addition, **a partial crash in the construction does not erase what the excursion contributed to the underlying state** (Sections 7.6.6 and 7.6.7), and it is this accumulation that leads toward the terminal crash. It is an open question whether repeated edge instabilities leave any comparable cumulative trace in the plasma that brings a terminal event closer, or whether the parallel is confined to the separate shapes of the two kinds of crash. It is equally open whether noise persistence plays a role in the timing of such events comparable to its role in Section 7.6.4.

A possible implication, should the structural parallel prove to be more than an analogy. If a genuine connection exists between the dynamics of tokamak plasmas and the system dynamics described in this section, the results of Section 7.6.4, read through the nested picture, point toward a possible stabilizing strategy along two lines. In the nested picture, the holding time of the generator corresponds to how long a nested structure keeps accelerating before the moving average detaches from it. In the simulation, **a faster generator, meaning a shorter holding time, produced**

smaller and less persistent fluctuations, and the singularity arrived considerably later. In nested terms, the excursions are cut short before they can build up to a large amplitude, and the completion of the outermost level, and with it the terminal crash, is postponed.

The first physical counterpart would be to **accelerate the plasma flow, in particular its sheared component**, so that fluctuations decorrelate more quickly. A related principle is already established in plasma physics on independent grounds: sheared flow is known to suppress edge turbulence by speeding up turbulent decorrelation (Biglari, Diamond & Terry, *Phys. Fluids B* 2, 1, 1990). The second counterpart concerns the partial crashes themselves: **triggering them more often than they would occur on their own, so that each has less time to build up.** This, too, has an established experimental counterpart. In ELM pacing by pellet injection, raising the ELM frequency above its intrinsic value was found to make the individual events milder (Lang et al., *Nucl. Fusion* 44, 665, 2004).

Within the present construction, however, **shorter excursions only delayed the singularity and made it rarer; they did not remove it entirely.** Full stability was reached only in the noise-free limit. The nested picture also indicates why: an excursion that is cut short still leaves its contribution in the underlying state, and these contributions continue to accumulate toward the completion of the outermost level. Should the parallel hold, such measures would therefore be expected to reduce the size of individual partial crashes, and to postpone a terminal one, more readily than to exclude it. Whether this correspondence reflects a shared mechanism, or only a structural resemblance, is left open here.

Both measures work in the same direction and share a limit, which the later parts of this work reach independently. Accelerating the decorrelation and injecting more frequently both distribute the intervention more evenly through the volume, and that is what their effectiveness rests on; **neither removes what has already accumulated where the intervention did not reach.** Part III, Section 1.1 arrives at the same conclusion from the pellet side, that the question is one of spatial coverage rather than of timing, and Part II, Section 4.1 gives the reason the coverage can never be complete: the intervention has a grain of its own, and that grain does not shrink as the quantity to be discharged shrinks. **The argument given here on dynamical grounds and the one given there on physical grounds meet at the same place**, which is some reason to take it seriously and none to take it as settled.

8. CONCLUSIONS

The outcome of this part is a negative one, and it is what the rest of the work is built on. A single observed quantity, however long the record and however well the parameters fit, does not settle what produced it: the exponent is not recoverable from the trajectory, the comparison with a power law does not discriminate, and the supply contraction that is Bitcoin's own contribution was shown in Section 3.1 to be insufficient on its own. **Bitcoin is therefore not a candidate for membership in the class described in Part II**, and nothing in what follows treats it as one. What it provided was the question. **Understood in that light the work is best read not as a price-prediction tool but as an attempt to build a measurement structure**, which makes it both narrower and broader than a curve fit. It is narrower in the sense repeatedly emphasized throughout: **the specific numerical parameters presented here describe one calibrated historical stretch, can be invalidated by a sufficiently extreme market event, and carry no claim to physical or economic necessity.** As Section 2.1 shows, even the qualitative structure they describe does not depend on these particular values.

It is broader in a different sense. The recurring structure of this paper's argument — first excluding Bitcoin's own supply contraction as a sufficient driver (Section 3.1), then separately excluding ordinary demand growth on the same grounds (Section 3.2) — points toward a more general framework than a single asset's price curve. What this exercise amounts to is the use of an asset with a hard, independently verifiable supply constraint as a fixed reference point, against which the behavior of everything on the other side of the price ratio can be examined. Read this way, **the contribution of this paper is not only a descriptive curve fit to one asset's history, but a candidate measurement instrument:** a fixed point against which the internal dynamics of an external, otherwise difficult-to-observe system might be studied indirectly, through the shape of the divergence it produces.

A direct corollary follows from this framing, worth stating explicitly, and Part II makes it more precise than it could be here. **What a price series reads is the relation between demand and a contracting free supply, and neither of those is the series itself:** in the vocabulary of Part II, Section 8.5, the price is an observation of a closing gap rather than the gap or what closes it. If the source of the feedback loop this paper's mathematics requires is, by the exclusions in Sections 3.1 and 3.2, external to Bitcoin's own supply dynamics, then no amount of further analysis confined to Bitcoin's own price and supply data could ever supply a positive derivation of that mechanism — only further exclusions of additional internal candidates, of the same kind already carried out. Confirming what the external process actually is would require studying that external system directly, on its own terms, which lies outside this paper's stated scope. **This is not a gap this paper could close by being more thorough within its own boundaries; it is a structural consequence of where the argument itself locates the missing term.**

This corollary is not merely theoretical. Exploratory attempts, in the course of preparing this paper, to detect a signature of this feedback mechanism directly within Bitcoin's own price statistics — momentum-style dependence between trend-relative price level and subsequent returns, and scale-invariance in the distribution of price fluctuations across different time horizons — did not yield supporting evidence, and in some cases pointed toward the opposite, ordinary market behavior. **This is consistent with, though it does not by itself prove, the structural conclusion above: if the mechanism is genuinely external, examining the instrument — or its wiring, Bitcoin's own supply and demand dynamics — should not be expected to illuminate it, and the negative results obtained here are compatible with exactly that.** This should not be read as a claim that no analysis of Bitcoin's own data could ever bear on the question; only that the specific attempts made here did not.

This paper does not establish this broader framing conclusively, and **it should be read as the direction the preceding sections point toward rather than a claim defended in full here**. It is offered as the most honest summary of what this body of work, taken as a whole, appears to be reaching for.

9. PLAIN-LANGUAGE SUMMARY

This paper compares a mathematical curve based on the tangent function, which contains a self-reinforcing feedback term, with a simpler alternative, the Power Law, on Bitcoin's historical price lows, in an attempt to detect whether such feedback is present. The tangent-based curve fits these lows more closely than the best possible Power Law fit on the same data, and the paper uses this fit to estimate an effective feedback exponent for that historical stretch. A simplified simulation also shows that the same overall shape can emerge from fluctuating, noise-driven exponents, without any finely tuned values. The curve has a mathematical feature, a point where its value becomes undefined, which is interpreted as a theoretical marker of possible extreme illiquidity, not as a price prediction. This is exploratory, descriptive work, not a forecasting tool, and it carries real uncertainties, discussed openly in Section 7. **Read together with the exclusions in Sections 3.1 and 3.2, this points toward a broader, more tentative possibility: that the model's greatest value may lie less in the price curve itself than in its potential use as an instrument for observing the structural pressures of the external system it points to** — a direction this paper proposes rather than conclusively establishes.

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Intermittent Superlinearity — A Blow-Up Theorem for Fluctuating Exponents

1. SCOPE AND CLAIMS

The earlier part of this work arrived, by way of a single asset's price geometry, at a construction that turned out not to depend on that asset at all. A state variable grows under a feedback term whose exponent is not fixed but fluctuates, and the system reaches a finite-time singularity even when the average feedback is not superlinear. That observation began as a numerical result. **It is established as a theorem in Section 3 of this paper**, and the remaining question, which is genuinely open, is whether anything in the physical world realizes the structure.

The object of study. Consider a non-negative state variable $y(x)$ governed by

$$\frac{dy}{dx} = k(1 + y^{m(x)}), \quad k > 0$$

where **the feedback exponent $m(x)$ is not a constant but a fluctuating quantity whose values straddle the critical boundary $m = 1$** . For a constant exponent the behavior is classical: a finite-time singularity exists if and only if $m > 1$. Section 2 sets out that case exactly, and Section 3 treats the fluctuating one.

A single instance, not the whole system. Everything stated in Sections 2 and 3 concerns **one instance of this equation**. For such an instance the growth rate never falls below $k > 0$, so **its state cannot decrease**: subcritical stretches slow it, but nothing reverses it (Lemma 1). **This monotonicity is a property of the single instance, not of the full dynamics**. In the construction this paper is ultimately concerned with, instances

of the same equation are **nested within one another**: faster instances ride on slower ones, and the observed quantity follows the innermost instance that currently exists. When a nested instance completes its singularity, it ceases to exist, and the observable **detaches from it and falls back onto the supporting instance beneath**. Seen at the level of the whole system, **the observable can therefore decline**, even though no single instance ever gives back any of its own growth. Such declines are bounded from below for as long as a supporting instance exists beneath them. **The outermost instance has no support beneath it**. If it, too, runs to its own singularity, there is no lower level for the observable to return to, and **the nested process as a whole terminates**. In the application that motivated this work, this corresponds to a breakdown of the logarithmic moving average itself, rather than a return to it (Part I, Section 7.6.6). None of this conflicts with the results below. The theorem of Section 3 requires only the monotone state of a single instance, and the declines described here occur at a different level, between instances rather than within one. **They are taken up once the nested construction is developed, after Section 4.**

Three claims of different status. They are kept apart deliberately, because they are not equally secure. **First, a proved result** (Section 3): the fluctuating case is not a perturbation of the constant case but a distinct regime, in which intermittent excursions above the boundary force the singularity almost surely, and in which the mean exponent plays no part in the outcome. **Second, a conjecture about the origin of the fluctuations** (Section 4): that they need not be imposed from outside, but may be produced by smaller, faster instances of the same equation nested within the system. **Third, a conjecture about the physical world** (Section 4): that the class defined here is not empty, that some real systems realize this structure. The third is the reason for the paper and the least settled part of it. **A demonstration that no such system exists would be an equally valid outcome.**

What is claimed, and what is not. This paper will use examples drawn from unrelated physical domains: edge instabilities in confined plasmas, nonlinear optical processes, accelerating creep in geomechanics, and noise-contaminated control loops that occasionally run away. **No claim is made that these share a physical or causal mechanism**, and nothing here belongs to solid-state physics, plasma physics or any unified physical theory. The claim is **mathematical isomorphism**: that the governing dynamics of these systems, once reduced, share the same structure, and that their behavior near the critical boundary is therefore governed by the same geometry of phase space.

This kind of claim is ordinary in dynamical systems theory. A pendulum and an LC circuit have no physical mechanism in common, yet the same second-order equation describes both, and consequences derived from that equation apply to both. **The shared object is the equation, not the hardware.** In the same sense, the systems examined here are treated as different physical realizations of one mathematical structure, each computing the same dynamics in its own substrate.

A note on terminology. The term "universality class" is used in this paper in a deliberately restricted sense: a set of systems sharing the same blow-up scaling exponent $1/(m-1)$ and the same qualitative response to fluctuations in m . **It is not used in the strict sense of critical phenomena**, where membership is established through renormalization group arguments and a full set of critical exponents. No such derivation is offered here, and none is claimed.

Order of argument. The mathematical case is developed first, and the physical examples are taken up only afterwards. This order is deliberate: the examples are tests of a stated structure, not the evidence from which the structure is inferred. Section 2 establishes the deterministic reference case, Section 3 proves the theorem and the criterion that follows from it, and Section 4 states what remains conjectural and how a candidate system is to be tested. **Where a candidate fails the test, this is reported as such.**

2. THE DETERMINISTIC SINGLE-FUNCTION CASE

Before any fluctuation is introduced, it is worth setting out precisely what can be said about **a single instance of this structure with a constant exponent**. Everything in this subsection is exact, elementary, and independent of the conjectures of Section 4. It serves as the reference case against which the fluctuating construction of Section 3 is compared.

The system. The structure consists of a **latent equation** governing an internal, dimensionless state variable, together with an **observation map** carrying that state into whatever quantity a given system actually exposes to measurement. Throughout this paper the observation map is taken to be linear, so that the observed quantity is the state itself expressed in the system's own units:

$$\frac{dy}{dx} = k(1 + y^m), \quad y(0) = 0 \quad f(x) = A \cdot y(x) / k$$

with $k > 0$, $A > 0$ and $m > 0$ constant. Here k carries the dimension of inverse x , A carries the unit of the observed quantity, whatever that happens to be, and m is dimensionless. Dividing by k is a normalization choice, made so that the observed quantity reduces to $A \cdot x$ in the early phase. A more general, monotone power map is discussed at the end of this section; **it changes none of the results below, and is set aside because the feedback mechanism itself does not depend on it.**

Proposition 0 (the rate is a clock, not a shape). Substituting $\xi = kx$ gives $dy/d\xi = 1 + y^m$, in which k no longer appears. The solution of the general system is therefore $y(x) = \tilde{y}(kx)$, where $\tilde{y}$ is the single universal solution of the reduced equation. Consequently **k is a pure rescaling of the**

horizontal axis and A a pure rescaling of the vertical axis: neither alters the shape of the trajectory. **With the observation map fixed, the family has exactly one shape parameter, m .**

Proposition 1 (existence and location of the singularity). Separating variables gives the exact distance to the divergence, measured from the origin:

$$x^* = \frac{1}{k} \int_0^\infty \frac{du}{1 + u^m} = \frac{1}{k} \cdot \frac{\pi / m}{\sin(\pi / m)}$$

The integral converges if and only if $m > 1$, which is the **blow-up criterion**: for $m \leq 1$ the trajectory is defined for all x and no singularity exists. Writing $G(m) = (\pi/m)/\sin(\pi/m)$, the result separates cleanly into $x^* = G(m)/k$: the exponent fixes the *dimensionless* distance to the singularity, and the rate fixes the clock. The closed form used in the earlier part of this work is the special case $m = 2$, where $G(2) = \pi/2$ and $x^* = \pi/(2k)$.

$G(m)$ decreases monotonically from infinity to one as m runs from 1 to infinity, so **the singularity can be pushed arbitrarily far out by approaching the critical boundary from above, but never closer than $1/k$, however strong the feedback**:

m	1.2	1.5	2	3	5	10
$k \cdot x^* = G(m)$	5.236	2.418	1.571	1.209	1.069	1.017

Proposition 2 (local scaling law). Near the singularity the constant term becomes negligible and $dy/dx \approx k y^m$, which integrates to

$$y(x) \approx [(m-1) k (x^* - x)]^{-1/(m-1)}$$

The blow-up exponent $1/(m-1)$ depends on m alone; k enters only through the prefactor. This exponent is the quantity by which two systems can be compared without reference to their units or substrates, and it is the candidate invariant on which the classification proposed in this paper rests.

Proposition 3 (remaining distance from a given level). The same integral taken from an arbitrary level y rather than from the origin gives what is still required to reach the singularity:

$$T(y) = \frac{1}{k} \int_y^\infty \frac{du}{1 + u^m} \approx \frac{y^{1-m}}{k(m-1)} \quad (y \gg 1)$$

Because $1 - m < 0$, this quantity **shrinks toward zero as the state grows**, and it is inversely proportional to k . This is the result from the deterministic case that carries the most weight for what follows: **the higher the rate k and the higher the level already reached, the shorter the interval of superlinear feedback needed to complete a singularity**. In the constant-exponent case treated here this is merely a restatement of Proposition 1. Its significance appears only once m is allowed to vary, since it then sets **how long an excursion above the critical boundary must last in order to be sufficient**, and that requirement weakens as the state grows.

Proposition 4 (the subcritical and critical cases). For completeness, since a fluctuating exponent spends part of its time at or below the boundary: for $m < 1$ the same separation of variables gives $y(x) \approx [(1-m) k x]^{1/(1-m)}$ at large x , **a power law with no singularity**, while $m = 1$ gives exactly $y(x) = e^{kx} - 1$, which is unbounded but finite at every point. **The three regimes are therefore power-law growth, exponential growth, and finite-time divergence**, separated by the single value $m = 1$.

The early phase. While $y \leq 1$, the feedback term satisfies $0 \leq y^m \leq 1$ for every $m \geq 0$, so the growth rate stays between k and $2k$ and the state grows at a nearly constant rate. **In this phase the trajectory carries almost no information about m** : systems that will diverge and systems that never will are, at this stage, indistinguishable from one another and from a feedback-free linear growth. The exponent becomes visible only once the state is large enough for the term y^m to dominate.

Roles of the parameters. The above fixes each one exactly. **k is the internal clock**: it sets how fast the system runs and nothing else, scaling the whole trajectory horizontally, so that x^* is inversely proportional to it. **m is the feedback strength**: it determines whether a singularity exists at all, its dimensionless distance $G(m)$, and the blow-up exponent $1/(m-1)$. Together, **k and m fix the singularity: m says how far in dimensionless terms, k says how long that is**. **A is the unit of measurement**: it converts the dimensionless state into the quantity a particular system exposes — a currency, a current, an intensity — and affects neither the shape of the trajectory nor the location of the singularity.

Remark: a non-linear observation map. Some systems expose not the state but a power of it, $f = A (y/k)^n$ with $n > 0$ constant. Since this map is monotone, **it can neither create nor remove a singularity**, and x^* is unchanged; it only reshapes the approach. Eliminating the state gives a closed equation in the observed quantity,

$$\frac{df}{dx} = n k A^{1/n} f^{1-1/n} + n k^{m+1} A^{(1-m)/n} f^E, \quad E = 1 + \frac{m-1}{n}$$

in which the first term alone is a plain power law and the second is the feedback term, with E the effective feedback exponent as it appears in the observed quantity. **$E > 1$ precisely when $m > 1$** , so the blow-up criterion survives the transformation unchanged, as it must. Two points are worth keeping from this. First, **only E is accessible from an observed trajectory, and E constrains just the combination $(m-1)/n$** : any pair with the same ratio produces the same observable, so the feedback exponent cannot be recovered from a record of the output unless the observation map is known independently. Second, with the linear map adopted here, $n = 1$ and $E = m$: the observed feedback exponent and the exponent in the equation coincide, which is why the general map adds nothing to the argument that follows.

2.1 A worked instance, and the origin of this construction

The simplest non-trivial case is $m = 2$, $k = 1$, $A = 1$ and $n = 1/2$, for which the latent state is $y = \tan(x)$ and the observed quantity is

$$f(x) = \sqrt{\tan(x)}$$

Differentiating directly gives $df/dx = \frac{1}{2} \tan(x)^{-1/2} \sec^2(x) = \frac{1}{2}(f^{-1} + f^3)$, which is exactly what the general expression above returns: the first exponent is $1 - 1/n = -1$, the second is $E = 1 + (m-1)/n = 3$, and both coefficients are $n k = \frac{1}{2}$. **Two points are worth drawing from this.** First, $\sqrt{\tan(x)}$ is *not* itself a member of the family $dy/dx = k(1 + y^m)$ for any m , since it produces two terms of different exponent; it is that family seen through a power map. **The same dynamic therefore presents as a different equation depending on the power through which it is read**, which is the concrete form of the identifiability limitation noted above: here a latent quadratic feedback appears in the observed quantity as a cubic one. Second, this particular function is where the present line of work began, before the latent equation was identified; it is recorded here so that the relationship between the two is explicit.

3. INTERMITTENT SUPERLINEARITY: A BLOW-UP THEOREM

This section states and proves the central mathematical claim of the paper. The structure of the argument is a race between two quantities established in Section 2: **the duration of superlinear feedback still required to complete a singularity, which shrinks as the state grows, and the probability of an excursion of that duration, which does not**. Once the first falls below a fixed window length, the second stops shrinking altogether, and the outcome follows.

Setting. Let y satisfy

$$\frac{dy}{dx} = k(1 + y^{m(x)}), \quad y(0) = y_0 \geq 0, \quad k > 0$$

where the exponent path $m(x)$ is a stochastic process satisfying:

(S1) Boundedness and regularity. $0 \leq m(x) \leq M < \infty$, with right-continuous sample paths. **(S2) A superlinear tail.** There exist $\mu > 1$ and $\Delta > 0$ such that the probability of a sustained excursion, $q = P(m(x) \geq \mu \text{ throughout a window of length } \Delta)$, is strictly positive. **(S3) Recurrence.** Conditional on everything that has happened before it, each window carries at least some fixed probability $q > 0$ of containing a sustained excursion above μ . **Independence is sufficient for this but not necessary:** a regenerative or finite-state Markov exponent process satisfies it, and so does any process whose conditional law of the next window is bounded below whatever the past. This is an assumption about the exponent process, not a consequence of the construction. **The nested case of Section 4 does not satisfy the independence form of it and does satisfy this one:** there the fluctuations of a level are produced by the levels within it, and the cascade is a piecewise-deterministic Markov process, Markov in its full state but not in any single level taken alone, so the windows of one level are separated by a slower state that carries information between them. **That dependence is permitted here**, and why the conditional bound survives it is set out after the theorem. Section 5 adds a second reason to be careful with it, since a bounded system has no unlimited supply of fresh trials. **Section 4.4 leaves open a question that bears directly on this point:** whether a completed instance, which is brief and enormous when measured on a slower level, can itself act as a fluctuation there. Were that so, the renewal a level cannot generate within itself would be supplied from below. **That is a question and not a result**, and nothing in the theorem depends on it; the theorem applies where (S3) holds, and says nothing where it does not.

Assumption (S2) is weak: if the paths are right-continuous and the exponent exceeds μ with positive probability at all, then it stays above μ for some positive duration, and some window length Δ therefore has $q > 0$. **Note what (S2) does not mention: the mean of m , which plays no role anywhere in what follows.**

Lemma 1 (ratchet). Since $dy/dx \geq k > 0$, the state is strictly increasing and $y(x) \geq y_0 + kx$. **Any level Y is therefore reached no later than $x = Y/k$, and is never given back.** Subcritical stretches slow the growth; they cannot reverse it. **This holds within a single instance; declines of the observable in a nested system arise between instances, not within one (Section 1).**

Lemma 2 (excursion criterion). Suppose $y(a) = Y \geq 1$ and $m(x) \geq \mu > 1$ throughout $[a, a + \tau]$. For $y \geq 1$ and $m \geq \mu$ we have $1 + y^m \geq y^\mu$, so comparison with $dz/dx = k z^\mu$, $z(a) = Y$, gives $y \geq z$ on the interval. Since z diverges at $a + Y^{1-\mu}/[k(\mu-1)]$, **the singularity is completed within the excursion whenever**

$$\tau > \tau^*(Y) = \frac{Y^{1-\mu}}{k(\mu - 1)}$$

Lemma 3 (threshold level). $\tau^*(Y) \leq \Delta$ as soon as

$$Y \geq Y_\Delta = [k(\mu - 1) \Delta]^{-1/(\mu-1)}$$

By Lemma 1 this level is reached by $x_\Delta = \max(1, Y_\Delta)/k$ at the latest. **Beyond that point, every window of length Δ in which the exponent stays above μ is sufficient on its own.**

Theorem (intermittent superlinearity). Under (S1):

(a) No blow-up without a superlinear tail. If $m(x) \leq 1$ for almost every x , then $1 + y^m \leq 2(1 + y)$ for all $y \geq 0$, hence $y(x) \leq (1 + y_0)e^{2kx} - 1$. **The trajectory is finite at every finite x , however long it is observed.**

(b) Almost-sure blow-up with one. If (S2) and (S3) also hold, then y reaches a finite-time singularity with probability 1. Partition $[x_\Delta, \infty)$ into consecutive windows of length Δ . By Lemma 3 each is sufficient, and by (S2) each succeeds with probability at least $q > 0$; by (S3) each attempt succeeds with conditional probability at least q given the past, so $\sum_n P(A_n \mid F_{n-1}) = \infty$ and Lévy's extension of the Borel–Cantelli lemma gives infinitely many successes almost surely; one suffices. **Independent attempts are the special case**, for which the probability of surviving the first N windows is at most $(1-q)^N$. **Moreover**

$$E[x^*] \leq x_\Delta + \Delta/q \qquad P(x^* > x_\Delta + N\Delta) \leq (1-q)^N$$

What the theorem says, in words. A system of this kind does not need superlinear feedback on average, nor even most of the time. **It needs only that superlinear feedback be possible, recurrent, and capable of lasting a fixed positive interval.** The mean exponent, which determines everything in the constant case, drops out entirely: it affects how quickly the state climbs, and therefore the timing, but not the outcome. The separating condition is not the average but **the upper tail of the exponent's distribution together with the recurrence of its excursions**, and part (a) shows this is sharp: remove the tail above 1 and the singularity disappears. Stated in one line, this is **the tail criterion: what decides the outcome is the upper tail of the feedback exponent, not its average.**

Why the requirement collapses. Writing $u = \ln y$ for $y \geq 1$ puts the mechanism in its clearest form, $du/dx \approx k e^{(m(x)-1)u}$. **An excursion of a given size above the boundary acts on the logarithm of the state with a strength that grows exponentially in that logarithm.** An excursion that is negligible early is decisive later, not because the excursion has changed, but because the state has.

Three predictions, and a numerical check. The theorem makes claims that can be tested directly against simulation. First, blow-up should occur for exponent processes whose mean is below the boundary. Second, the distribution of blow-up times should have an exponential right tail, from the geometric bound above. Third, **the timing should be controlled by how far above the boundary the exponent reaches, more than by anything else.** In a two-state exponent process with equal switching rates and $k = 0.05$, integrated numerically over 25 runs per setting, the median blow-up time behaves as follows at a fixed mean of exactly 1:

Upper state μ	1.1	1.2	1.4	1.8
Median x^*	286	159	80	44

Every run in every setting reached a singularity, including settings with a mean exponent of 0.76. **A separate check varied the persistence alone, holding the marginal distribution fixed, and found only a weak effect:** across switching rates spanning a factor of eighty, the median moved by a few percent. This is worth stating plainly, because it revises a reading suggested by the earlier part of this work, where more persistent noise appeared to bring the singularity forward. **In that construction, slower fluctuations also had larger amplitude, and the theorem indicates that the amplitude, not the persistence, was doing the work.** Persistence matters through (S2), by making q positive at a usable window length; beyond that threshold its influence is secondary.

Proposition 5 (prefactor noise is insufficient). The theorem is specific about *where* the fluctuation has to act. Suppose instead that the exponent is fixed at or below the boundary, $m \leq 1$, and that the noise enters the rate, so that $dy/dx = k(x)(1 + y^m)$ with $k(x) \leq k_{\max} < \infty$. Comparison with the

constant-rate equation at k_{max} gives $y(x) \leq (1 + y_0)e^{2k_{max}x} - 1$ by part (a) of the theorem. **No finite-time singularity occurs, however large, however persistent and however heavy-tailed the fluctuation in the rate.**

This is elementary, but it is the sharpest practical tool the theorem provides, and it applies directly to candidate systems. **A fluctuation that scales the growth rate, shifts the curve, or modulates the gain cannot produce this behavior; only a fluctuation that reaches the exponent of the feedback term can.** In examining any physical system, therefore, the first question is not how strong its noise is, but which of the two it perturbs. Many systems whose time series show sharp, intermittent build-up-and-collapse events are excluded at this step, and the exclusion is decidable from the governing equations rather than from the appearance of the data.

What the theorem does not establish. Assumption (S3) is not decorative. If the exponent is drawn once and then frozen, a realization below the boundary never blows up, and no amount of waiting helps; the result requires that the system keep drawing. The bound on $E[x^*]$ is also crude, since it credits the system with nothing during subcritical stretches and only counts whole windows; in the numerical checks above, blow-up arrives far earlier than the bound allows. **Most importantly, the theorem assumes the fluctuations as given.** It says nothing about where they come from, and in particular it does not yet cover the nested case, where the excursions are produced by smaller instances of the same equation. That case introduces a dependency the theorem does not handle: **finer levels run on faster clocks, so their excursions are not only smaller but shorter**, and whether their durations shrink faster than $\tau^*(Y)$ does is precisely the question Conjecture 2 of Section 4 turns on. That comparison, and the conditions a candidate system must satisfy, are taken up in Section 4.

The nested construction satisfies the recurrence condition under one further restriction, and this is worth setting out since it was the chief doubt about the theorem's reach. In that construction only the fastest level draws on an external sequence, and the levels beneath are driven by completions above them, so the windows of any one level are plainly not independent. **They do not need to be.** Two features supply the conditional bound directly. The external sequence at the top is drawn afresh in each window, so whatever has happened, the probability that it contains a sustained excursion is unchanged. And the supporting level never decreases, so by Proposition 3 the excursion duration still required never increases: **a window that would once have sufficed always suffices thereafter**, and the past can only improve the prospects of the future, never worsen them.

What this leaves is a condition on the coupling rather than on independence. If each completion above delivers an excursion whose duration shrinks as the level rises, and shrinks faster than $\tau^*(Y)$ does, then q is not bounded below and the argument fails. **That is precisely the mechanism by which the plateau of Section 5.1 forms**, and it is the same comparison Conjecture 2 of Section 4.4 turns on. **The recurrence condition is therefore not an obstacle peculiar to nesting**, but the same question the construction already poses, arriving in a second place: whether what the finer levels deliver fades faster than the requirement they are meant to meet.

The condition and its consequence, stated compactly. Let A_n be the event that the n -th window contains a sustained excursion above μ , and let $\mathcal{F}_{n-1}$ denote everything that has occurred before it. Assumption (S3) is

$$P(A_n \mid \mathcal{F}_{n-1}) \geq q > 0 \quad \rightarrow \quad \sum_n P(A_n \mid \mathcal{F}_{n-1}) = \infty \quad \rightarrow \quad P(A_n \text{ infinitely often}) = 1$$

by Lévy's extension of the Borel–Cantelli lemma. **The step that matters is the first:** the bound is on the conditional probability, so the windows may be as dependent as the system makes them, provided the dependence never drives that probability to zero. **Independence is the special case in which the conditioning may be dropped**, and it gives the sharper tail bound $P(\text{no success in } N \text{ windows}) \leq (1-q)^N$ quoted above.

3.1 When the Feedback Becomes Undetectable

Section 2 established that trajectories with different exponents are hard to tell apart while the state is small. **A particular choice of fluctuation makes the difficulty worse: it produces a trajectory that follows a power law almost to the end, shows no detectable feedback anywhere in that range, and terminates in a singularity regardless.** The figure below is one such run. **It was produced with the simulation of Part I, Section 2.2**, which compares a deterministic superlinear trajectory, a fitted power law and a noise-driven one, and which serves here unchanged; **only the two exponent values and the probability between them were altered**, so the code is not repeated.

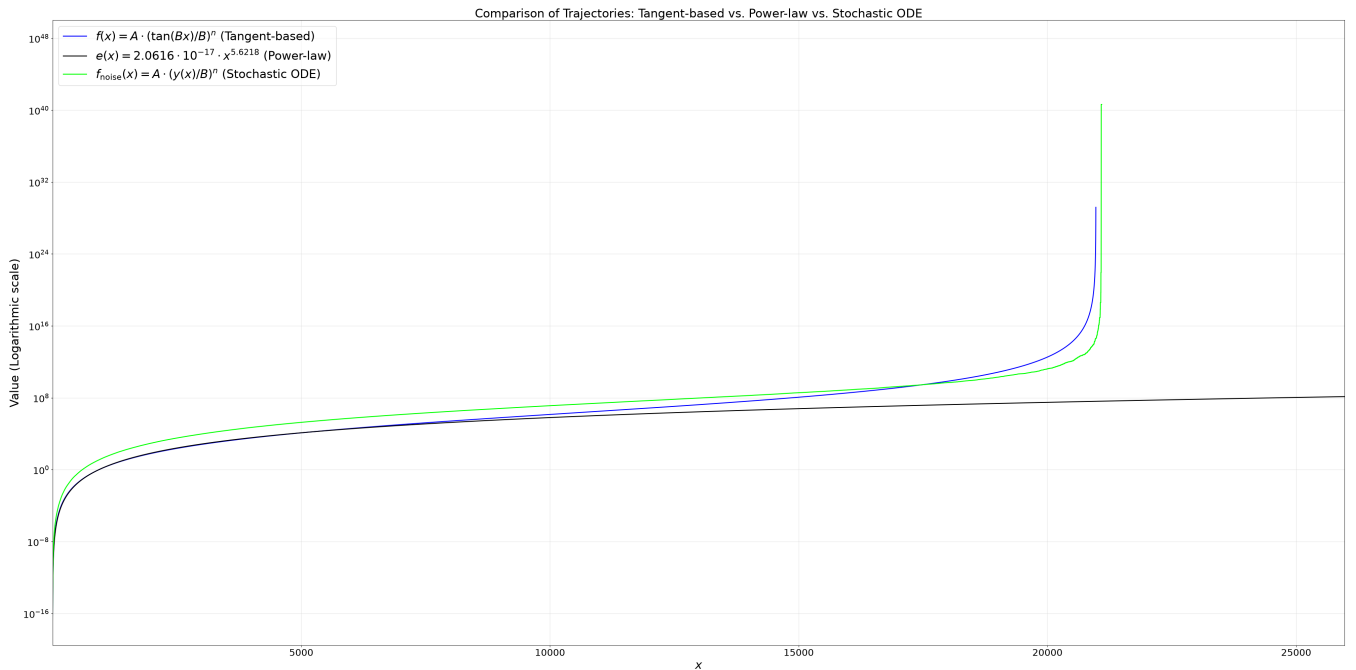


Figure 0b.

The mechanism is elementary. Suppose the exponent is drawn between a high supercritical value and one far below the boundary, here 2.75 and 0.1, with the subcritical draw much the more common. During a subcritical stretch the feedback term is $y^{0.1}$, which is 1.58 at $y = 100$ and 3.98 at $y = 10^6$: **over six orders of magnitude of the state it barely moves.** The equation then reduces to

$$\frac{dy}{dx} \approx 2k \quad \rightarrow \quad y \approx 2kx \quad \rightarrow \quad f = A(y/B)^n \propto x^n$$

which is a power law, and not merely something resembling one. Its exponent is n , the exponent of the observation map, and this is why the agreement in the figure is exact in form rather than a coincidence of scaling: **a subcritical stretch deep enough to flatten the feedback term leaves the observation map as the only thing shaping the trajectory.** In the run shown the two curves coincide closely because the map's exponent and the fitted power-law exponent happen to be near one another, which is a property of that particular calibration rather than of the mechanism. **While the exponent spends most of its time far below the boundary, the system behaves as though it had no feedback at all,** because a feedback term raised to a small power is almost a constant.

The flattened trajectory sits above the deterministic one rather than below it, and this is not an inconsistency. While $y < 1$ the ordering of Proposition 6 reverses: a small exponent gives a *larger* feedback term there, so the subcritical draws accelerate the early trajectory instead of retarding it. The effect is bounded, since the growth rate stays between k and $2k$ whatever the exponent, but the observation map raises that factor to the power n and the result is visible on a logarithmic plot. **What matters is that a vertical offset is a constant factor, which a power-law fit absorbs into its coefficient.** Feedback is detected from the curvature of a record and not from its level, and it is the curvature that the subcritical stretches remove.

The singularity arrives nonetheless. The mean exponent in the run shown is 0.63, well below the critical value, and the supercritical draws are rare. They are also violent: at $y = 100$ the term $y^{2.75}$ exceeds 10^5 , so the few excursions accomplish what the many quiet stretches do not. The result is the theorem of Section 3 in an extreme form: **the mean is not merely uninformative about the outcome, it points the wrong way.**

What this means for any attempt to detect feedback from a record, stated plainly. The general method is to compare a candidate trajectory against a feedback-free baseline and to read an effective feedback exponent from the difference; the comparison carried out in Part I, Sections 3.3.1 and 6.3, is one instance of it, and the membership tests proposed in Part III rest on the same idea. **This run shows a case in which that comparison would find nothing.** Over the observed range the trajectory is a power law to within the resolution of any such test, the effective feedback exponent measured from it would be indistinguishable from unity, and the conclusion drawn would be that no superlinear feedback is present. The conclusion would be wrong. **Failure to detect feedback therefore does not exclude a finite-time singularity,** and a negative result from the comparison of Part I, Section 6.3 should be read as a statement about what the data can show rather than about what the system will do.

What it does not show. The parameters were chosen to produce this effect and are not claimed to describe anything. The masking depends on the subcritical draws being far below the boundary rather than merely under it: as the two exponent values are brought closer together, the measurable feedback returns, and the trajectory separates from the Power Law earlier and more visibly. **Nor does the timing of the singularity distinguish the two cases here:** in the run shown it falls close to the deterministic one, and where the fluctuation is fast the averaging leaves little variation between runs. **Timing is a weak discriminator under fast noise, and the shape of the early record is no discriminator at all.**

The effect just described is not a curiosity of the parameters chosen, and it has a definite direction. The following proposition states it, and the proof is elementary.

Proposition 6 (fluctuation accelerates, and masks only downward). Let the exponent be drawn independently at each step from a fixed distribution with mean $\bar{m}$ and non-zero variance, and compare the resulting trajectory with the deterministic one at a fixed exponent m_0 . Then, for $y > 1$:

(a) Fluctuation always accelerates. Writing the exponent as $m = \bar{m} + \delta$ with δ of zero mean,

$$E[y^m] = y^{\bar{m}} \cdot E[e^{\delta \ln y}] \geq y^{\bar{m}}$$

by Jensen's inequality applied to the convex function e^u , with equality only at $y = 1$. **A fluctuating exponent therefore reaches its singularity sooner than a fixed exponent equal to its mean**, and the gain grows with the variance. This is convexity in the sense of Taleb and Douady: the system benefits from variability in the quantity to which it is convex, here the exponent.

(b) Matching the singularity time requires a lower mean. If the fluctuating trajectory is arranged to reach its singularity at the same point as the deterministic one, then by (a) its mean exponent must satisfy $\bar{m} < m_0$. The variance has already supplied part of the drive, so less is left for the mean to supply.

(c) The early record is then weaker, never stronger. Expanding the moment generating function for small $\ln y$ gives $E[e^{\delta \ln y}] = 1 + \frac{1}{2}\text{Var}(m)(\ln y)^2 + \dots$, a correction that is quadratic in $\ln y$, while the deficit in the mean enters linearly as $y^{\bar{m}} = y^{m_0} e^{-(m_0 - \bar{m}) \ln y}$. **Near the origin the linear deficit dominates the quadratic gain**, so the fluctuating trajectory carries a weaker feedback term than the deterministic one over the early range, and the two cross only later. **The crossing exists if and only if the upper reach of the exponent exceeds m_0** , since otherwise the fluctuating term is below the deterministic one everywhere.

The practical content of this is a one-way bound. A fluctuating exponent can make the early record look flatter than the deterministic case, and arbitrarily so: the deeper the subcritical excursions, the flatter it looks, which is the masking demonstrated above. **It cannot make the early record look steeper** while still arriving at the same singularity. **An observed early trajectory therefore places a lower bound on the feedback, not an upper one:** whatever superlinearity is visible in it is at least present, and more may be hidden, but a trajectory cannot display more feedback early than the deterministic case it matches.

Two conditions, both of which matter. The first is that the exponent's distribution is fixed: it must not depend on the state or on how far the process has run. **If that is relaxed, the bound fails and the record can be made to show anything**, since an exponent held high early and low afterwards reaches the same singularity while displaying strong feedback at the start. What rules this out is stationarity, not the construction. The second is the restriction to $y > 1$. Below it the inequality reverses, but there the feedback term lies between 0 and 1 whatever the exponent and the growth rate stays between k and $2k$, so the difference is a constant factor rather than a change of shape and does not affect the exponent an observer would measure.

3.2 The Theorem Describes an Undisturbed System

The theorem states what a system does when left to the dynamics described. **It says nothing about a system that is acted on from outside**, and this needs stating separately, because the omission is easy to read as a defect in the theorem rather than as a limit on its scope.

The point is best made by an objection. Consider two gravitating bodies, idealized as points, approaching in otherwise empty space. The separation closes in finite time, and the collision frequency or the speed diverges with it. **One might object that the theorem cannot apply, since nothing prevents the bodies from being pushed apart:** a rocket, a collision with a third body, a field imposed from elsewhere, and the approach is reversed. **The objection is correct and does not bear on the theorem**, because a system that is pushed apart is no longer the system the theorem describes. The hypotheses specify the evolution; they do not claim that no other influence can exist in the universe, and no theorem about an evolution equation could.

The same point arises wherever the surroundings are not empty. An evaporating black hole loses mass at a rate that would carry it to zero in finite time; immersed in ambient radiation it may absorb more than it emits and grow instead. **The sign of the net rate is then set by the surroundings**, and which of the two descriptions applies is a question about the environment rather than about the equation. The same is true of a landslide whose slope is drained, a plasma whose heating is reduced, a market in which an authority intervenes, and every other case in this paper: **in none of them does intervention refute anything, and in all of them it replaces the system under discussion with a different one.**

What this costs is real and should not be minimized. A record produced by a system that was acted on combines the dynamics with the intervention, and the two are not separable without knowing the intervention. **This is not a limitation of the framework but a limit on what any record of an intervened system can be asked to show**, and it applies with equal force to a deliberate control action, to a change in operating conditions, and to an unrecorded external input. Where a candidate system is routinely managed, as a plasma or a reactor is, the record available for testing is a record of the management as much as of the dynamic.

A separate confusion is worth distinguishing from this one. "The theorem does not apply" can mean two quite different things: that the system is being acted on from outside, or that the system has no fluctuating exponent to begin with. **Only the second is a statement about the system itself**, and the two should not be run together, since a case may be set aside for either reason or for neither.

On the gravitational example this paper takes no position. The exponent attributed to it in Section 6.2 is derived from the inverse-square relation for point masses and has not been measured, which places it in the same category as every other derived exponent here and subject to the same caution. **Nor is it obvious that it could be measured on the system it describes:** real bodies are extended rather than point-like, and what an instrument reports of a pair of extended bodies is not the quantity the derivation concerns. Whether the underlying interaction is continuous at every scale, or whether it resolves at some scale into discrete events of the kind that govern the mechanical arrangements of Section 6.1, is a question this work has no means to address. **The example is therefore retained for the arithmetic of the relation and not as a case that has been decided either way.**

4. IDENTIFYING A CANDIDATE SYSTEM

The theorem of Section 3 is a statement about an equation. **Whether any real system is governed by such an equation is a different question**, and so is the question of where the fluctuations in one would come from. This section begins with what can be said with reasonable confidence: how a singularity can appear in a physical system at all, what a candidate system has to satisfy, and what can be resolved by measurement. **The conjectures follow afterwards, and are marked as such.**

4.1 Singularities in a physical system

In the physical world, no quantity actually becomes infinite. A singularity is therefore a property of an equation, not of a measurement. In a system described by such an equation, the divergence may be completed mathematically, but **the observed quantity has to leave it at some point:** what is recorded ends, changes character, or falls back to whatever remains beneath it. In the simulations of this work the same thing appears as the point at which the function stops being defined and its value becomes NaN. **The detachment is not an anomaly of the model but the form in which a singularity can appear in a physical system at all.**

The singularity is a criterion rather than a prediction, and it is worth saying so before the physical cases. What it decides is whether the Osgood integral converges, and that single question separates the two regimes on which everything in this paper rests: the critical value of the exponent, the theorem of Section 3, and the membership conditions of Section 4.2. **It is not a claim that any observed quantity reaches infinity**, and nothing in the framework requires one to.

This matters because the divergence is the part a measurement never sees. The observed quantity detaches from the structure beneath it before the approach is complete, often long before, and in a bounded system it may detach while the record still looks unremarkable. **The content of the distinction therefore lies in the approach and not in the endpoint:** a subcritical trajectory never accelerates, a supercritical one does, and that difference is present throughout the trajectory rather than only at its end. **What happens at the point itself is outside what this work can say**, and Section 9 leaves it open.

What cuts the divergence off need not act at the level where measurement takes place. The level at which an equation applies and the level at which data is recorded may differ, so a divergence may be completed at the former and appear only as a bounded event at the latter. A system may also reach values that are effectively meaningless from the point of view of the observer, whether because no measurement remains possible, or because the quantity loses the interpretation it had over the range in which it was defined. **In such cases the distinction between a completed divergence and an extremely large finite value is not decidable by measurement**, and nothing in this work depends on deciding it.

Whether anything in nature diverges without limit is a separate question. **The expansion of the universe and its long-term future are the obvious place where a genuine infinity might be at stake**, and that question lies outside the scope of this work.

One form of regularization recurs often enough to be worth stating in advance. A whole family of realizations, including every one constructed in Section 6, is driven by a **closing gap**: two surfaces, two thresholds, or two levels of any kind, between which something travels, and whose separation shrinks. The observed quantity in such a system is a frequency, which is a speed divided by that separation, and it diverges as the separation goes to zero. **The separation cannot go to zero.** Beneath it there is always a scale set by the thing moving within it, and below that scale the gap is not merely small but undefined. In the circuit of Section 6 this is the resolution of the comparators, below which it is no longer determined which threshold is higher; between two surfaces it is the size of the body bouncing between them; in a gas it is the diameter of the particle. **The singularity is therefore not prevented by something acting against it, but by the disappearance of the quantity that was to diverge.**

This gives the regularization of a physical realization a definite location rather than a vague one. Where a candidate system is driven by a closing gap, the question "what truncates the divergence?" has a first answer that can be stated before any mechanism is investigated: the gap closes to the scale of its own contents and no further. The event that terminates the process occurs there, and the observed quantity reaches whatever value corresponds to that scale, which is finite and in principle calculable. **Any further truncation, by saturation, dissipation or loss, acts earlier than this one and is specific to the system; this one is not. The remark applies only where a closing gap is present**, which is a

wide class but not every candidate: a system in which the diverging quantity is a stress, a density or an accumulated damage has no gap and must be examined on its own terms. Where there is one, however, the location of the cutoff is known in advance of any investigation of the mechanism.

What happens at that point is a redistribution rather than an ending. When the gap reaches the scale of the body within it, the body no longer traverses the gap, and the motion that was being counted ceases. The energy associated with it does not. It passes into the internal degrees of freedom of the body itself, which in a mechanical case means deformation and internal oscillation, and those have their own characteristic frequencies, set by the body's size and by the speed at which disturbances travel through it. **The observed quantity collapses while the energy does not**, and an observer counting the original events records a termination where the system has undergone a transfer.

Two things follow, and the second is a limitation. The first is that this offers one reading of why systems of this kind change character at close range rather than simply stopping: the interaction that dominates at separations large compared with the bodies is not the one that dominates when they are comparable, and the change of regime is a change in what the degrees of freedom are, not only in the strength of a force. **The second is that nothing here says the redistributed energy enters another instance of the same construction.** That would require a closing gap at the new scale and a non-decreasing quantity driving it, and neither follows from the transfer. **The observation is that the energy reappears elsewhere; where it goes afterwards is outside what this construction can say**, and the open question of Section 9 concerns exactly that step.

The construction has a second limit, at the other end, and the same quantity sets both. Near the origin the feedback term is negligible and the state grows at the constant rate k , so it emerges from a fluctuation of width σ only after a distance of order σ/k . **The start of a run is therefore as inaccessible as its end**, and for the same reason: the noise does not shrink with the quantity being measured. What differs is the consequence. At the upper limit the boundary absorbs, so a fluctuation that reaches it ends the run, which is the mechanism of Section 6.3. **At the lower limit there is nothing to absorb**, and the noise merely displaces where the run appears to begin.

The origin also carries an apparent singularity that belongs to the observation map rather than to the state. With $y \approx kx$ near zero, the observable behaves as $f \propto x^n$, so for $n < 1$ its slope is infinite there while the state passes through smoothly. **The function from which this work began is an instance:** $\sqrt{\tan(x)}$ has $n = 1/2$ and rises vertically from the origin, where $\tan(x)$ itself, with $n = 1$, passes through it without difficulty. **Nothing in the dynamics distinguishes the two;** the difference is entirely in how the state is read, which is the distinction Section 4.3 draws throughout.

Neither limit bears on the outcome, and both bear on what can be measured. A run observed from a finite distance above its origin and truncated a finite distance below its asymptote is the only kind of run available, and the inaccessible portions at each end are set by the same fluctuation. **Where that fluctuation comes from is the question of Section 7.1**, and it is the same question at both ends.

4.2 Membership conditions

For a system to fall within the scope of this paper, five conditions must hold. A system that fails any of them does not belong to the class, however suggestive its phenomenology. **They are not equally easy to check, and it is worth saying in advance which is which.** Conditions 4.2.3 and 4.2.4 are settled by measurement: the first is usually immediate, and the second asks for simultaneous records from different parts of a system, which many fields already collect. **Condition 4.2.5 is settled by reduction rather than by measurement**, since where a transfer acts is read off the governing equations once they are written in the form of Section 1. **Conditions 4.2.1 and 4.2.2 are the hard ones**, and for a reason this paper has established rather than merely encountered: a counter may exist without being countable in practice, and by Section 4.3 no measurement of a completed event can establish how its exponent behaved beforehand.

For those two, an argument carries the same weight as a measurement, and nothing else is available. What satisfies them is an account: a mechanism within the system by which the quantity accumulates irreversibly, or by which something reaches the exponent and recurs. **A simulation can serve as part of such an account where it is built from the field's own description rather than from this one.** A model of a candidate system, constructed from what that field already holds about its mechanics and run without reference to the construction here, which then exhibits the behaviour described, is evidence of a kind that a trajectory fitted after the fact is not. **The requirement is that the model not be built to produce the answer**, which is the same requirement of independence that Condition 4.2.4 places on derived quantities.

A practical consequence is an order of work. Condition 4.2.4 excludes fastest and costs least, so a candidate is best taken there first; 4.2.5 next, since it needs only the equations; and the remaining two last, where the answer will be an argument and the honest report is often that the question is open. **None of this weakens the conditions:** a candidate that cannot be shown to satisfy 4.2.1 and 4.2.2 is not thereby a member, and the position to report is that it is undetermined. Establishing membership requires reducing the system's own governing equations to the form of Section 1, not observing a visual resemblance in its time series.

A third route is available where the conditions cannot be fully settled, and it should be used with the qualification that follows. Where a candidate satisfies what can be checked and the rest remains open, the construction can be applied provisionally, as a working description, and judged by what it then yields. **If it directs attention to something the field was not looking for, and that thing is found, the case for membership is strengthened:** an accumulating quantity not previously identified, an anticorrelation not previously measured, a drift in a control parameter nobody had plotted. **This is the ordinary way an account earns its place**, and there is no reason to hold this one to a different standard.

The qualification is that the prediction must be one the construction could have got wrong. A framework applied to a system it was chosen to fit will usually explain that system, and the more freedom it has the more surely it will; Section 3.1 shows how far a trajectory can be reshaped by parameters alone. **What counts is a statement the established description does not make and would not expect,** checked against data not used in forming it. **An account that explains what is already known explains nothing,** however well it does so, and a success of that kind should not be reported as evidence of membership.

Conditions 4.2.1 to 4.2.5 describe a nested system, and 4.2.6 is an alternative to them rather than an addition. A candidate may qualify by the first route, as a structure of coupled instances, or by the second, as a single instance whose exponent is accounted for from within. **The two routes do not share all their requirements.** The non-decreasing state of 4.2.1 and the restriction to stretches of a history in 4.2.3 apply either way. **Condition 4.2.5 has no content for an isolated instance,** since it governs transfers between levels and there are none; and 4.2.4 is usually untestable there, since it compares parts of a system against one another. **What the second route asks for instead is set out in 4.2.6,** and it is a heavier requirement than any of the first five.

One further remark bears on whether a candidate is worth taking through the conditions at all. The construction describes a single scalar gap, and the realizations in this paper are spatially extended objects: an accumulator holds charge throughout its volume, a winding occupies a path, and neither is a point. **They reduce to one number nonetheless, and the reason is worth naming:** the charge within a conductor redistributes far faster than the circuit changes, so the whole object is at one potential and a single value describes it. The reduction is not an approximation imposed for convenience but a consequence of a separation of timescales that happens to hold.

This suggests where to look before attempting a reduction. Where a system contains a quantity that equilibrates within its parts much faster than those parts evolve, a scalar description is available and the conditions above can be applied to it. **Where no such quantity exists, and the state varies across the system as fast as it changes in time, there is no single gap to identify,** and a candidate of that kind is not excluded but requires a different treatment, one this paper does not provide. **The question is therefore not how many dimensions a system occupies but whether something in it equilibrates quickly enough to stand for the rest,** which is the same kind of condition as 4.2.2 and is settled the same way.

4.2.1 Condition

A state variable that does not decrease over the regime considered, **at the level of a single instance of the equation.** This condition does not extend to the observable of a nested system. The observable may decline when it detaches from a completed instance and falls back onto the supporting instance beneath, and **the process as a whole may stop** if the outermost instance also ends in its singularity and the observable detaches from it as well (Part I, Sections 7.6.6 and 7.6.7). **Nor does a non-decreasing state imply an observable that grows without bound:** in physical realizations, the approach to a singularity presumably ends in an abrupt transition and a detachment rather than in a literal infinity, although systems not bound by a physical budget, such as a market price, may prove exceptions.

The condition is easier to satisfy than it looks, because the variable need not be a quantity anyone measures. In a nested system the levels complete, reset, and complete again, and nothing one can point to rises monotonically: each level's state falls back whenever it is reset. **What does not fall back is the count of completions.** Writing $N_i(x)$ for the number of instances of level i that have completed by x , and $c_i > 0$ for what each delivers to the level beneath, the quantity

$$S(x) = \sum_i c_i N_i(x)$$

is non-decreasing by construction, since every term is a counter and counters do not run backwards. **This is the variable the condition is about.**

The circuit of Section 6 shows the counter plainly, although it carries only one instance. There the events are the oscillator's own cycles rather than the completions of a level beneath: each cycle delivers a fixed charge q to the accumulator through the winding, so after N cycles

$$V(x) = V_0 + N(x) q/C \quad G(x) = G_0 - N(x) q/C$$

The oscillator frequency rises and falls many times in the course of this, since the oscillator capacitor is discharged at every cycle, and the accumulator voltage never falls at all. **The observable oscillates and the state ratchets,** and a counter relates the two. **This is the single-instance case:** the cycles are driven by the oscillator rather than by completions of a faster structure, so the sum above has one term.

Joining two such circuits would give the nested form, and nothing in the above changes. Suppose the faster circuit, instead of charging its own accumulator, delivers its charge into the accumulator of a slower one, so that each completion there closes the slower gap. **The slower instance then has no oscillator of its own:** what drives it is the faster structure, through a winding or a current source, with the faster circuit's controller deciding when the delivery occurs. The sum acquires a second term, the state of the slower instance is driven by the count of the faster one's completions, and the slower gap closes to its own singularity in the same way. **Only the first circuit in such a chain needs an external fluctuation,** which is the arrangement the simulation of Section 4.3 implements and which Conjecture 2 of Section 4.4 concerns. The description is given here because the counter is the same object in both cases.

Three things follow, and the third is the one that matters for identifying candidates. First, **the weights need not be known.** Whether S is non-decreasing depends only on the c_i being positive, not on their values, so a candidate can satisfy this condition without anyone determining how much each level contributes. Second, **the weights are needed only to estimate the exponent**, which Section 4.3 shows is not recoverable from a record in any case. Third, **the condition reduces to a question with a yes-or-no answer:** does the system contain events that are irreversible, countable in principle, and cumulative in their effect? **That is a far weaker requirement than exhibiting a monotone measurement**, and it is the form in which the condition should be applied.

Two cautions. The weights must not change sign: a coupling that sometimes removes what another adds breaks the construction, and S is then not a ratchet but a random walk. **And countable in principle is not countable in practice:** in a rock mass the events are microfractures, which are neither individually observable nor exhaustively detectable, so the condition may hold while S remains permanently unmeasured. **This is a condition on what the system does and not on what an instrument can see**, which is the same separation drawn in Section 4.3 and in Section 4.2.2.

Two things in that arrangement are worth making explicit, since neither is obvious from the description. The first is the hierarchy. On completing, the faster instance is reset to the level of the slower one beneath it, so it always begins its next run from wherever the slower state has reached. **This is the ordering of Section 4.3 realized in hardware**, and without it the two circuits would simply run side by side; the reset is what makes one the support of the other.

The second is that the exponent of the slower instance is not set by anything in the slower circuit. Because the faster instance restarts from the slower gap, its time to completion is proportional to that gap, so its completions arrive at a rate $v \propto 1/G$. Each delivers a fixed charge, so

$$\frac{dG}{dt} = -\frac{q}{C} \quad v \propto -\frac{1}{G} \quad \rightarrow \quad G \propto (t^* - t)^{1/2} \quad \rightarrow \quad m = 3$$

The nesting therefore produces an exponent rather than merely a timescale, and it produces the same value the winding gives in the single-instance case, for the same reason: the delivery is counted per event. **And because the faster instance carries an external fluctuation of its own, its completion times fluctuate**, so the rate at which the slower gap closes fluctuates with them, and the slower exponent is not a constant. **That is the structure of the simulation of Section 4.3 in hardware:** an exponent generated by the level above rather than imposed, and fluctuating because that level fluctuates.

The winding is one way to carry the delivery and not the only one, and the exponent follows from how the delivery is metered rather than from the component. A current source switched on for a fixed interval at the start of each cycle delivers $q = I t_{on}$, a constant per event, and gives $m = 3$ exactly as the winding does. **The same source left running continuously delivers per unit time instead and gives $m = 2$** , which is the distinction of Section 6.2 appearing between the levels rather than within one.

The two are the ends of one arrangement, and a real circuit travels between them. As the frequency rises the period shortens, and once it approaches t_{on} the delivery windows meet: the source is on for most of each cycle, the per-event description fails, and what the accumulator sees is a continuous current. **The effective exponent therefore drifts from 3 toward 2 in the course of a run**, and the crossover lies near $f \approx 1/t_{on}$, which is known in advance. The winding behaves the same way for the same reason, with the additional frequency dependence of Section 8.

In the nested arrangement the per-event rate itself comes from the reset. An instance restarting at the level beneath takes a time proportional to that level's gap, which is what makes the completions arrive at a rate $v \propto 1/G$; were it to restart from zero, they would arrive at a constant rate and the exponent would be 2 whatever the delivery. **Both ingredients are therefore required**, and neither belongs to the component chosen to carry the charge.

4.2.2 Condition

A feedback term whose effective exponent fluctuates across the critical boundary, with a non-vanishing probability of sustained excursions above it. **This is required of the system over its run and not of every stretch of it.** An exponent may be fixed, or fixed above the boundary, over any portion of a trajectory without bearing on the condition: by Section 4.3 the terminal approach of a completed event is supercritical in every case, since nothing else completes one, and by Section 5.2 a physical system is pushed toward 2 in that stretch whatever it did earlier. **A measurement showing a settled exponent somewhere is therefore not a failure of this condition**, and a candidate is not excluded by one. The exponent must also vary on a timescale short compared with the one over which the state grows appreciably, while its individual excursions still persist long enough to matter. **This is a ratio and not a rate**, and the condition should not be read as requiring the fluctuation to be fast in absolute terms. **The construction has no timescale of its own:** by Proposition 0 the rate parameter is a pure rescaling of the horizontal axis and leaves the behaviour otherwise unchanged. A system whose exponent moves over decades while its state builds over centuries satisfies this condition exactly as well as one whose exponent moves in microseconds while its state builds in milliseconds. **A candidate should therefore not be excluded for being slow, nor admitted for being fast**, and the comparison is always between two quantities belonging to the same system.

This condition is stated of the system, not of a measurement of it, and the distinction is not a formality. By Section 4.3 and Section 3.1 neither the record nor the exponent fitted to a completed event can establish whether it holds: any completed event was completed during an excursion above the boundary, so a fit returns a supercritical value whatever the distribution behind it, and deep subcritical stretches flatten the early record toward a power law rather than revealing themselves. **A candidate therefore cannot satisfy this condition by exhibiting a measurement, and cannot fail it by exhibiting one either.**

What the condition asks for is an account of the exponent. It is satisfied when the governing equations of a system, reduced to the form of Section 1, show a feedback exponent that is not a constant, together with an identified quantity capable of moving it across the boundary and a reason why that quantity recurs. **A derived exponent does not refute it.** Where a reduction yields a fixed exponent, what has been shown is that the idealization from which it was derived has a fixed exponent, which is a statement about the derivation and not about the system; the caution of Part III, Section 1.1 applies in both directions. **Where neither an account nor a measurement is available, the condition is undetermined rather than unmet,** and a candidate in that position is an open case rather than a rejected one.

The separation of timescales is not a fixed property of a system, and this changes how the condition should be applied. By Proposition 3 an instance of rate k starting from a support Y completes after $Y^{1-m}/[k(m-1)]$, so the rate required for an instance to complete within a window of length Δ is

$$k \geq k_{\min}(Y) = \frac{Y^{1-m}}{\Delta(m-1)}$$

and this threshold falls as the support rises. An instance too slow to matter early in a run becomes fast enough later, not because anything about it has changed but because the level it rests on has risen. **The hierarchy therefore recruits levels as it proceeds,** and its depth at any moment is set by how much of the available range of rates lies above $k_{\min}(Y)$.

Two consequences, and the second is visible in the figure of Section 4.3. First, a candidate should not be excluded because the separation of timescales is unconvincing in its quiet phase: the condition asks for levels fast enough relative to the state's growth, and which levels those are depends on where the state has got to. **Second, the terminal rise is steeper than a fixed hierarchy would produce.** The counter of Section 4.2.1 gains terms as new levels begin completing, so the gap closes faster than the contribution of any fixed set of levels would close it. In the figure this appears as the simultaneous densification of two levels rather than one as the slower instance approaches its end. **The exponent has not changed; the number of contributors has,** and a fit to that stretch would report the difference as a drift.

4.2.3 Condition

Physical systems of this kind are finite, so the dynamics need only hold over stretches of their history rather than throughout it. The early spread of a virus or of a bacterial population is an example: such a phase can show superlinear feedback and the blow-up character described here, while the system as a whole remains bounded by the resources available to it. Because of that bound, **the observed quantity cannot pass a certain level;** it leaves the instance it was following and falls back to a lower one, which may be zero. What is required is therefore not that a system behave this way throughout its history, but that at least one such stretch be identifiable. **The same system may also behave differently at different scales:** feedback found at one scale need not be present at another, and where it is present, its exponent need not be the same.

4.2.4 Condition

Measurements of different parts of the system must be anticorrelated. Where the accumulating quantity is extensive and the parts exchange with one another, Section 4.3.2 shows that the correlation between any part and the remainder is exactly -1 , and between two parts $-1/(N-1)$. **This is a consequence of the exchange and not an assumption,** so a candidate in which simultaneous measurements at different points rise and fall together, rather than against one another, is not transferring anything between them and does not have the structure this construction describes.

The condition is useful because it is cheap to test and hard to satisfy by accident. It requires two or more simultaneous records from different parts of the same system, which many fields already collect, and it asks of them something a system driven by a common external influence would fail: such a system moves its parts in the same direction. **It says nothing about the shape of any single record,** which by Section 4.3 may be smooth, flat, or indistinguishable from a power law, and it is therefore independent of every other condition here.

Two qualifications. The anticorrelation is exact only where the total is conserved over the interval examined; where the system as a whole is also gaining from outside, the measured correlation is the sum of a negative exchange term and a positive common term, and the first may be masked. **And it holds between a part and the rest, which is what an instrument compares;** between two small parts of a large system it is weak, falling as $1/(N-1)$, so a test on two arbitrary sensors in a large system may find nothing even where the structure is present.

Separately, the record of any one part is not expected to be smooth or to fit closely. The dynamics are driven by instances that end in singularities, so a record resolving them contains abrupt rises and the falls that follow; and since the exponent is driven by noise, repeated runs under identical parameters differ substantially in the timing and size of individual events. **What can be compared with data is the character and the scaling of a trajectory, not the position of any particular excursion.**

The condition is about the conserved quantity and not about whatever is easiest to measure. What is transferred between parts is seldom what an instrument reports; the reported quantity is often a consequence of the transfer, and a consequence is under no obligation to be anticorrelated with anything. **Two records moving independently, or one moving while the other stays still, do not fail this condition**, since a part that has received without yet responding shows nothing at all. **Applying the test to the wrong quantity gives an answer about that quantity.**

The quantity need not be measured directly, and in most systems it cannot be. A derived quantity serves, provided the derivation is independent of the hypothesis being tested: the anticorrelation is then between computed values, one falling as the other rises, and the test is performed on quantities neither of which was observed. **Which derivations qualify is a matter for the field concerned**, since it turns on what else is known about the system and on what can be computed from it without assuming the structure at issue.

The requirement of independence is the one to watch. A computed quantity is only as good as the model that produced it, and a model built to describe transfer will produce transfer. **The derivation must come from somewhere this construction does not reach**, and where no such route exists the condition cannot be tested rather than being unmet. **That is a common position among the candidates of Part III**, and it should be reported as such rather than as a failure.

4.2.5 Condition

The fluctuation must reach the exponent of the feedback term, not merely its rate. **This applies to the couplings within the structure:** a transfer from one level to the level beneath that moves only the rate there cannot produce a singularity at that level, whatever its size, which is Proposition 5 and which has decided several cases in this work.

It does not apply to whatever drives the lowest level. What a cascade requires at its base is a recurring source of events, and the character of that source is unconstrained: in the circuit of Section 6 it is an ordinary oscillator, and a switch closed at irregular intervals would serve equally.

Whether the base source reaches an exponent or a state variable is a separate question, taken up in Section 7.1, and a candidate is not excluded by failing to identify an exponent-driver at its base.

4.2.6 Condition for the Blow-Up Theorem

An isolated instance can be a member without satisfying anything about nesting, and the route to establishing it is different. A single system matching the theorem directly, with no structure beneath it, is admissible: the theorem concerns an exponent process and says nothing about where the process comes from. **What such a case must supply instead is an account of the exponent itself**, since there is nothing above it to furnish the fluctuation, and that account has to come from within the system.

In practice this means a derivation rather than a measurement. What would satisfy the condition is a reduction of the system's own equations showing that its feedback exponent is a quantity rather than a constant, together with an identified mechanism moving it and a reason it recurs. **No candidate has yet been brought this far.** The systems considered in Part III are nested or are not yet shown to be anything; the one case in this paper where an isolated instance has a well-understood exponent is the circuit of Section 6, and there the exponent is fixed by construction rather than fluctuating. **The requirement is demanding, and deliberately so**, since without the hierarchy there is nothing else to appeal to.

A caution attaches to such cases specifically. An isolated instance admits routes to termination that are not the theorem's, as Section 6.3 shows for a subcritical circuit whose gap carries noise: there the exponent never moves and the run ends regardless. **A termination therefore establishes nothing about the exponent in an isolated system**, which is why the account has to be given in advance rather than inferred from the outcome.

4.3 Resolvability

The conditions of the preceding subsections are easier to state than to picture. This subsection therefore sets out a simulation of the construction, not as evidence for anything, but as **an illustration of the distance between an idealized nested system and the quantity an observer of it would actually record**. The code follows; what it does, and what it does not, is described first.

What is simulated. Four instances of the equation of Section 1 run simultaneously, with rates differing by a factor of ten at each step, so that each level is ten times faster than the one supporting it. **Only the fastest level draws on an external exponent process**, taken at each step from two values straddling the critical boundary; the levels beneath it receive their fluctuations from the completions of the level above, through a coupling of adjustable strength and duration. **That coupling is a simplification rather than a model of anything:** the completion of an instance is made to raise the exponent beneath it for a fixed interval, which stands in for a transfer whose actual form is unknown. When any level exceeds its cutoff, that level and every level above it are reset to the value of the level beneath, which is the construction's own account of an instance completing its asymptote and the structure above it falling back onto its support. The lowest level resets the entire system. **The observable is not any of these levels:** it is a moving average taken over the logarithm of the fastest one, and it is drawn separately.

Environment. The simulation was run on Fedora Workstation under JupyterLab, in Python 3, using NumPy for the state arrays and the exponent draws, pandas for the rolling average, and Matplotlib for the figure. It requires no other dependency and runs in a few seconds.

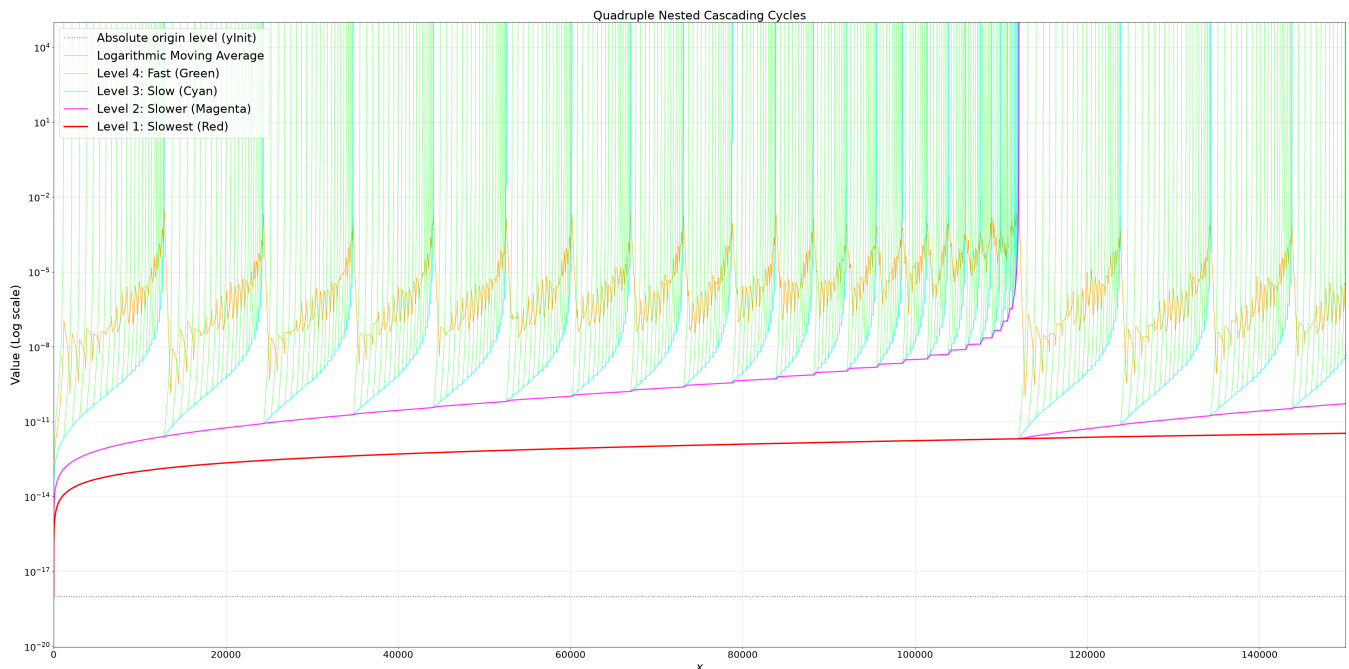


Figure 3a.

Ten points about what this does and does not represent.

1. It shows the unbounded case. The system has no ceiling: each level runs to its cutoff and the cascade continues indefinitely. Real systems are bounded in the sense of Section 4.2.3, and nothing here addresses the terminal behaviour of a finite structure. **What it does illustrate, and what it is included for, is the measurement problem,** which is present whether or not the system is bounded.

2. The observable is a logarithmic moving average, and it is not faithful. This is the point of the figure. Even here, where the underlying structure is exactly known and perfectly nested, **the average tracks none of the four levels:** it runs above the supporting structure, below the excursions, and it climbs steeply in the approach to a large event because the excursions beneath it are growing, not because the support is. **A single moving average is offered here as an illustrative device and not as a good readout;** a faithful reading would presumably require several, related to one another as the levels are, which is a conjecture rather than a result.

3. The nesting is idealized. Each level is an exact instance of the same equation, differing only in rate, and each resets exactly onto the level beneath. **A real system organized this cleanly is close to inconceivable,** and the figure should be read as a diagram of a relationship rather than as a model of anything.

4. Both sources of fluctuation are present, and they enter at different places. Only the fastest level draws on an external exponent sequence; the levels beneath it receive their fluctuations from the completions of the level above, through a coupling whose strength and duration are set by parameters. **The two are different claims and should not be conflated.** An externally supplied sequence is an assumption: the fluctuations are given, and the theorem of Section 3 takes them as given. **The coupling is Conjecture 2 of Section 4.4,** which holds that the fluctuations need not be imposed at all, and it is implemented here so that the conjecture can be examined rather than only stated. **Nothing in the theorem depends on it,** and the external sequence alone, driving every level, is the configuration the theorem describes. In the construction this paper proposes, the fluctuation at a given level would come from the faster instances running within it, or from the system's fluctuations generally, and would therefore differ from level to level and be correlated with what the faster levels are doing. **The rates and the other parameters would likewise vary rather than remaining fixed.** The cutoffs are of the same character: they are numerical safety limits chosen to keep the arithmetic finite, standing in for physical mechanisms that would truncate a real divergence, and the simulation marks those steps as missing values so that the plotted line breaks instead of drawing a connection that did not occur.

5. It is not deterministic. The exponent is drawn at random at every step, so **no two runs are identical.** The fluctuation is fast relative to the growth, so the differences between runs are modest in appearance, but the timing of individual events is not reproducible and should not be read as if it were. **The mean exponent in the configuration shown is slightly below the critical boundary,** and the cascade nevertheless proceeds, which is the theorem of Section 3 in visible form.

6. Every parameter is exposed, and the illustration is meant to be varied. The seed of the generator, the two exponent values and the probability between them, the four rates, the cutoffs, the window of the moving average, the plotted range, and the number of levels can all be changed; so can the structure itself, by making the cutoffs dynamic, by giving each level its own exponent process, or by weighting the coupling between levels. **Reading the figure is less instructive than changing one parameter and watching what moves,** which is the reason the code is given in full rather than described.

7. Because the levels are dynamically coupled, the exponent of the system as a whole may not be a well-defined quantity. This is worth stating because it bears on every measurement proposed in this paper. When the completion of a faster instance acts on the exponent beneath it,

and the instances complete often enough that their effects overlap, **the exponent in force at any moment reflects the coupling parameters rather than any property the system has on its own**; and a measurement taken away from that regime returns something else again. **The shape of the record is correspondingly unstable under the parameters.** One setting produces a smooth, rounded rise of the kind a textbook would draw; another produces a long flat stretch ending in a rise that appears almost vertical; the configuration shown in Figure 3a lies between the two. **A reader who varies the code will see this immediately, and should take it as the point rather than as an artifact.**

8. The moving average and the event rate do not fail in the same way. As the figure shows, the moving average of the observed level bears little relation to any of the levels present. **The rate at which events occur does not share that defect:** it rises as the supporting level rises, in the manner described in Section 5, and it does so whichever level is examined. Where a choice exists between reading a level and counting events, the second is the more informative, and the simulation can be used to check how far that holds under each setting.

9. How far the moving average follows a level depends on the shape of the rise. Where a level rises gradually, the average tracks it to a higher point before detaching; where the approach to the asymptote is abrupt, the average leaves it almost at once. **The point of detachment is therefore not a property of the averaging window alone**, and the same window applied to two differently shaped runs of the same system will appear to behave differently.

10. The cutoffs must be ordered, and this is an artifact of the simulation rather than a feature of the construction. Each level terminates at a numerical ceiling, and those ceilings are set higher for the faster levels than for the slower ones. **No such ordering means anything in a system with genuine mathematical singularities**, where every instance diverges without bound and there is nothing to order. It is required here because the arithmetic must stay finite and because a level has to be seen to complete before the level beneath it can respond. **The consequence is that the figure shows events at each level separated by many orders of magnitude in amplitude**, which is a property of the chosen ceilings and not of the dynamics; reading anything from the relative heights of the spikes at different levels would be reading the cutoffs.

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

# --- 1. PARAMETERS ---
A = 10e-17
B = 10e-5

k1 = 0.1 * B      # Level 1 (slowest)
k2 = 1 * B        # Level 2 (slower)
k3 = 10 * B       # Level 3 (slow)
k4 = 100 * B      # Level 4 (fast)

yInit = 1e-6      # Starting value for all levels

windowM4 = 700    # Moving average window for L4
plotWindow = 150000 # Adjustable observation window on the x-axis

# Exponent (m) parameters for each level
m1Base = 0.85      # Level 1 (Red) - default base (< 1)
m1Boost = 1.3      # Level 1 (Red) - boosted exponent (> 1)
redBoostSteps = 10000 # Duration of boost for Red when Magenta starts

m2Base = 0.85      # Level 2 (Magenta) - default base (< 1)
m2Boost = 1.3      # Level 2 (Magenta) - boosted exponent (> 1)
magentaBoostSteps = 300 # Duration of boost for Magenta when Cyan starts

m3Base = 0.85      # Level 3 (Cyan) - default base (< 1)
m3Boost = 1.3      # Level 3 (Cyan) - boosted exponent (> 1)
cyanBoostSteps = 30 # Duration of boost for Cyan when Green starts

# Noise configuration for Level 4 (Green)
p = 0.3
numSteps = max(50000, plotWindow + 2000)

# --- 2. NOISE GENERATION (Only for Level 4 / Green) ---
m4Noise = np.random.choice([1.2, 0.8], size=numSteps, p=[p, 1-p])

# --- 3. SOLVING THE DIFFERENTIAL EQUATIONS ---
y1 = np.zeros(numSteps)
y2 = np.zeros(numSteps)
y3 = np.zeros(numSteps)
y4 = np.zeros(numSteps)

y1[0] = yInit
y2[0] = yInit
y3[0] = yInit
```

```

y4[0] = yInit

y1Visible = np.zeros(numSteps)
y2Visible = np.zeros(numSteps)
y3Visible = np.zeros(numSteps)
y4Visible = np.zeros(numSteps)

y1Visible[0] = yInit
y2Visible[0] = yInit
y3Visible[0] = yInit
y4Visible[0] = yInit

# Boost timers for Red, Magenta, and Cyan
redBoostTimer = redBoostSteps
magentaBoostTimer = magentaBoostSteps
cyanBoostTimer = cyanBoostSteps

for i in range(1, numSteps):
    # Dynamic exponent for Red (L1)
    if redBoostTimer > 0:
        m1 = m1Boost
        redBoostTimer -= 1
    else:
        m1 = m1Base

    # Dynamic exponent for Magenta (L2)
    if magentaBoostTimer > 0:
        m2 = m2Boost
        magentaBoostTimer -= 1
    else:
        m2 = m2Base

    # Dynamic exponent for Cyan (L3)
    if cyanBoostTimer > 0:
        m3 = m3Boost
        cyanBoostTimer -= 1
    else:
        m3 = m3Base

    m4 = m4Noise[i-1]

    # Calculate increments based on previous step
    dy1 = k1 * (1 + y1[i-1]**m1)
    dy2 = k2 * (1 + y2[i-1]**m2)
    dy3 = k3 * (1 + y3[i-1]**m3)
    dy4 = k4 * (1 + y4[i-1]**m4)

    y1Temp = y1[i-1] + dy1
    y2Temp = y2[i-1] + dy2
    y3Temp = y3[i-1] + dy3
    y4Temp = y4[i-1] + dy4

    # --- L1 Function (Red) Singularity Check ---
    if y1Temp > 1e50:
        # Absolute reset of the entire system to yInit
        y1Visible[i] = np.nan
        y2Visible[i] = np.nan
        y3Visible[i] = np.nan
        y4Visible[i] = np.nan

        y1[i] = yInit
        y2[i] = yInit
        y3[i] = yInit
        y4[i] = yInit

        # Every level resets and restarts -> trigger all boost timers
        redBoostTimer = redBoostSteps
        magentaBoostTimer = magentaBoostSteps
        cyanBoostTimer = cyanBoostSteps

    # --- L2 Function (Magenta) Singularity Check ---
    elif y2Temp > 1e40:
        y1[i] = y1Temp
        y1Visible[i] = y1[i]

        y2Visible[i] = np.nan

```

```

y3Visible[i] = np.nan
y4Visible[i] = np.nan

# Reset everything above to the L1 level
y2[i] = y1[i]
y3[i] = y1[i]
y4[i] = y1[i]

# Magenta restarts -> boosts Red; Cyan and Green restart as well
redBoostTimer = redBoostSteps
magentaBoostTimer = magentaBoostSteps
cyanBoostTimer = cyanBoostSteps

# --- L3 Function (Cyan) Singularity Check ---
elif y3Temp > 1e30:
    y1[i] = y1Temp
    y1Visible[i] = y1[i]

    y2[i] = y2Temp
    y2Visible[i] = y2[i]

    y3Visible[i] = np.nan
    y4Visible[i] = np.nan

    # Reset L3 and L4 to the L2 level
    y3[i] = y2[i]
    y4[i] = y2[i]

    # Cyan restarts -> boosts Magenta; Green restarts -> boosts Cyan (Red timer unaffected)
    magentaBoostTimer = magentaBoostSteps
    cyanBoostTimer = cyanBoostSteps

# --- L4 Function (Green) Singularity Check ---
elif y4Temp > 1e20:
    y1[i] = y1Temp
    y1Visible[i] = y1[i]

    y2[i] = y2Temp
    y2Visible[i] = y2[i]

    y3[i] = y3Temp
    y3Visible[i] = y3[i]

    y4Visible[i] = np.nan

    # Reset L4 to the L3 level
    y4[i] = y3[i]

    # Only Green restarts -> Cyan boosted (Magenta and Red timers unaffected)
    cyanBoostTimer = cyanBoostSteps

# --- Normal Operation (No Singularities) ---
else:
    y1[i] = y1Temp
    y1Visible[i] = y1[i]

    y2[i] = y2Temp
    y2Visible[i] = y2[i]

    y3[i] = y3Temp
    y3Visible[i] = y3[i]

    y4[i] = y4Temp
    y4Visible[i] = y4[i]

# --- 4. OBSERVABLE QUANTITIES AND MOVING AVERAGES ---
f1Visible = A * (y1Visible / B)
f2Visible = A * (y2Visible / B)
f3Visible = A * (y3Visible / B)

f4 = A * (y4 / B)
f4Visible = A * (y4Visible / B)

absoluteBaseLevel = A * (yInit / B)

# Moving average for L4 (Orange)

```

```

logF4 = np.log(f4)
logM4 = pd.Series(logF4).rolling(window=windowM4, min_periods=1).mean()
M4 = np.exp(logM4)

# --- 5. PLOTTING ---
plt.figure(figsize=(40, 20))

plt.axhline(absoluteBaseLevel, color='gray', linestyle=':', linewidth=2, label='Absolute origin level (yInit)')

# Plot the four layers from top to bottom
plt.plot(M4[:plotWindow], color='orange', linewidth=1, label='Logarithmic Moving Average')
plt.plot(f4Visible[:plotWindow], color='lime', alpha=0.4, linewidth=1.5, label='Level 4: Fast (Green)')
plt.plot(f3Visible[:plotWindow], color='cyan', alpha=0.6, linewidth=2, label='Level 3: Slow (Cyan)')
plt.plot(f2Visible[:plotWindow], color='magenta', alpha=0.8, linewidth=2.5, label='Level 2: Slower (Magenta)')
plt.plot(f1Visible[:plotWindow], color='red', alpha=1.0, linewidth=3, label='Level 1: Slowest (Red)')

plt.yscale('log')
plt.title('Quadruple Nested Cascading Cycles', fontsize=25)
plt.xlabel('$x$', fontsize=25)
plt.ylabel('Value (Log scale)', fontsize=25)
plt.xlim(0, plotWindow)
plt.tick_params(axis='both', which='major', labelsize=20)
plt.legend(loc='upper left', fontsize=25)
plt.grid(True, alpha=0.3)

# Limit y-axis upper bound for cleaner visualization (increase if simulating larger values)
plt.ylim(top=1e5)

plt.tight_layout()
plt.show()

```

JupyterLab code for Figure 3.

Stated formally, the cascade is a piecewise-deterministic Markov process in the sense of Davis (1984): deterministic flow between events, jumps at events. Let $i = 1, \dots, N$ index the levels from slowest to fastest, with rates $k_1 < \dots < k_N$. The continuous state is $y = (y_1, \dots, y_N)$ and the discrete state is the vector of exponents currently in force; between events the flow is

$$\frac{dy_i}{dx} = k_i (1 + y_i^{m_i}), \quad i = 1, \dots, N$$

Two kinds of jump interrupt it. A *boundary jump* occurs when some y_i reaches its cutoff Y_i ; that level and every level above it are reset to the value of the level beneath, $y_j \leftarrow y_{i-1}$ for $j \geq i$, with y_0 the origin level, and the exponent of level $i-1$ is raised by a fixed amount for a fixed interval. A *random jump* redraws the exponent of the fastest level from its two-point distribution. **The first is state-triggered and carries the coupling; the second is the external fluctuation.**

The ordering of the levels is a property of this system rather than an assumption. The region $y_1 \leq y_2 \leq \dots \leq y_N$ is forward-invariant: the reset map places the upper levels exactly on the one beneath, and wherever two levels coincide the faster has the larger rate and separates upward. **Which instance supports which is therefore decided by the rates alone**, whatever the order in which the instances began.

4.3.1 What the Figure Shows

Four observations follow from the figure, and they are the reason it is included.

1. A measured exponent above the critical value does not exclude a system from this class, and the simulation shows why. This deserves to be stated separately, because the opposite assumption is natural and this paper has made it elsewhere.

Take the second-slowest level of the figure as the system under study, and suppose its code were unknown. **Its base exponent is 0.85, below the critical value**, and for most of its life it is subcritical and merely creeping. **It nevertheless ends in a singularity, and the exponent measured from that ending is not 0.85.** The completion happens during one of the intervals in which the level above has raised its exponent, so a fit to the terminal approach returns the raised value, above the boundary. **An observer with access only to the record would conclude that the exponent is supercritical and would be right about the ending and wrong about the system.**

This is not an artifact of these parameters but a consequence of Section 3. A singularity is completed only during an excursion above the critical value, since no subcritical stretch can complete one. **The exponent governing any completed event is therefore drawn from above the boundary by construction, whatever the distribution it came from.** It follows that fitting the terminal approach of a completed event can never return a subcritical exponent, in any system of this kind, and that the absence of a measured value below 1 is not evidence of anything.

The consequence for candidate systems is direct, and it reverses a test this paper has used. Where a measured exponent lies above the critical value, two readings remain open: a fixed supercritical exponent, as in the deterministic case of Section 2, or a fluctuating one whose mean lies below the boundary and whose excursions carry it above. **The measurement does not separate them**, and in a nested system, where the fluctuations come from the level above, the second is what the construction of this paper predicts. A candidate should therefore not be set aside on the grounds that its exponent has been measured above 1, which is the only value such a measurement could have returned.

What does separate the two has to be looked for elsewhere. Three places, in order of how much they ask of the data. **The scatter between events:** a fixed exponent gives a reproducible time to completion from a given level, a fluctuating one does not. **Excursions that did not complete:** a fluctuating exponent produces accelerations that begin and are not sustained, which a fixed one does not, and these leave a record in the interval before the event rather than during it. **The behaviour before the final approach:** by Section 3.1, deep subcritical stretches flatten the early record toward a power law, so the early trajectory of a fluctuating system looks less like feedback than the deterministic case it matches. **All three require the slow part of the record, which is also the part usually discarded**, and this is the same conclusion reached in Part III, Section 1.1 from a different direction.

A settled exponent in the terminal stretch is therefore expected rather than anomalous. Two separate arguments give it: a completion occurs only during a supercritical excursion, so the value governing it is drawn from above the boundary whatever the distribution behind it; and by Section 5.2 a physical system with a driver of bounded capacity is pushed toward 2 in that same stretch, since nothing else is deliverable there. **A record showing a stable supercritical exponent at the end of an event is consistent with a fluctuating exponent whose mean lies below the boundary**, and the condition of Section 4.2.2 asks about the system over its run rather than about any stretch of it.

2. The observed minima are not monotone, even in an unbounded system. The simulation has no ceiling, and its levels rise without limit; one might therefore expect the moving average to be pushed steadily upward as well. It is not. **In several places a later minimum of the average lies below an earlier one**, within the same stretch of the same supporting level. The reason is that the average is a reading of the fastest level and not of the support beneath it, so where it falls to depends on where the excursions happen to be when the structure above it resets, and not only on how far the support has risen. **This is worth stating explicitly because the opposite is a natural assumption:** that a system whose underlying state never decreases must produce a record whose troughs never decrease either. The figure shows that it need not, and it shows this in the simplest possible case, with the structure exactly known.

3. The quantity that moves the exponent need not be the quantity that is observed. What has been discussed so far concerns the relation between a record and the level it reads. There is a second gap, and it lies earlier: the exponent of the feedback is moved by some physical quantity, and there is no reason for that quantity to be the one an instrument happens to report. **Dynamic quantities are the natural candidates**, and each field has its own: an accumulated mechanical stress, an evolving current profile, the entanglement of a field configuration. Whichever it is must be identified within the field concerned, and this work cannot do it in general. **It can be done in one case.** In the circuit of Section 6 the observed quantity is the oscillation frequency, $f = v/G$, where v is the rate at which the oscillator capacitor is charged and G the gap between the thresholds. **The two are moved by different parts of the circuit, and the separation is physical rather than notional.** The rate is set by the current source feeding the oscillator; the gap is closed by whatever charges the accumulator, and the law by which that happens is what fixes the exponent, as the five cases of Section 6.3 show. In the mechanical analogue of Section 6.1 the accumulator is the approaching surface and the oscillator is the ball: **here the surface cannot act on the speed of the ball at all**, since the two are driven by separate sources, so a coupling delivered into the accumulator reaches the exponent and nothing else. **Where an intervention is delivered therefore decides which of the two it acts on**, and in this circuit the two channels are distinct components rather than two aspects of one quantity. **The practical consequence is that a membership test may require two quantities rather than one:** the observed level, and the quantity suspected of moving the exponent, examined together rather than separately. A single channel may show the events clearly and carry no information at all about what produced them. It is entirely possible for a discrete event to register strongly in one quantity, such as a jump in power, while what it leaves behind, such as a persistent structural change or a residual stress, registers in another and only loosely with the first.

4. The nested construction does not require the theorem, and is not an illustration of it. Every level in the figure runs the equation of Section 1, and nothing in the nesting depends on the exponent crossing the critical value. **A cascade whose levels all carry a fixed exponent above the boundary behaves in the same way:** each instance completes, the level above it falls back onto the one beneath, the support rises, and the next instance begins from a higher point. That version is deterministic, its completion times are reproducible, and the simulation produces it by removing the noise and leaving the rates.

What the theorem adds is a claim about which systems can do this, not about what the cascade does. Where the exponent is fixed above the boundary, the construction describes a system that would have terminated anyway and whose nesting only organizes the terminations. Where the exponent fluctuates across the boundary, the theorem says the terminations occur even though the mean is subcritical, which is a stronger claim about a wider class. **The two cases are distinguishable in the code and not from the figure**, which is the subject of observation 1 above.

The masking of Section 3.1 is likewise independent of where the mean lies. It requires only that the exponent fluctuate and that its subcritical excursions be deep, and both are available to a system whose mean is comfortably above the boundary. **Such a system can therefore show a flattened early record while remaining supercritical throughout**, and the flattening is not evidence that the mean has fallen below the critical value. **Taken with observation 1, this leaves the measured exponent uninformative in both directions:** a value above the boundary does not exclude a subcritical mean, and a record flattened toward a power law does not establish one.

4.3.2 Sampling a Part of the System: Anticorrelation Between the Part and the Whole

The preceding subsection concerned aggregation in *time*: over what interval a quantity is averaged. **A separate difficulty arises from aggregation in space**, that is, from measuring part of a system rather than all of it. The two are independent, and the second is the more counterintuitive, because it allows a correctly measured quantity to fall while the system it belongs to is rising.

The condition is about the whole. Where the accumulating quantity is extensive, so that the state of the system is the sum of the states of its parts, the non-decreasing requirement of Section 4.2 applies to that sum. **It does not apply to any individual part**, and nothing in the construction says it should. A sum can increase while some of its terms decrease, and if the parts exchange with one another, some of them must.

The elementary case, where the effect can be computed exactly. Take N units sharing a conserved total, $\sum_i x_i = X$, exchanging among themselves. Since the total is fixed, the variance of the sum vanishes:

$$\text{Var}(\sum_i x_i) = N \sigma^2 + N(N-1) \text{Cov}(x_i, x_j) = 0 \quad \rightarrow \quad \text{Cov}(x_i, x_j) = - \frac{\sigma^2}{N-1}$$

The correlation between any two parts is negative, and it is negative by necessity rather than by accident. For two parts the correlation is $-1/(N-1)$, which is weak when the parts are many. **Between one part and the sum of all the others it is exactly -1**, since the conserved total makes each the complement of the other: whatever one gains, the rest loses. **It is this second comparison, and not the first, that a measurement actually performs**, since an instrument reports a region and the quantity of interest belongs to the whole. This is the standard result for a conserved sum, and it is the reason a sampled portion is not merely a noisy copy of the whole: **it is an anticorrelated one**. In a system of a few hundred units the effect is weak for any single pair and considerable between a part and its complement.

A gas makes the effect concrete. The speed of one molecule in a container of N molecules varies erratically under collisions, while its long-time average approaches the ensemble average, and the time required for that convergence grows as the sample shrinks. Within a short window, one molecule may be observed slowing while the gas as a whole is being heated; over a long enough window it cannot. **What is measured here is not an error but a true statement about a part**, and the only mistake available is to read it as a statement about the whole. **The example is used here for the arithmetic of sampling alone**, and no position is taken on whether a gas, or any part of one, falls within the class described in this paper; that question is independent of the point being made.

The case that matters is when the exchange is not symmetric. Where every unit is equivalent, the anticorrelation is spread evenly and a sample is unbiased on average. Systems in which a process concentrates somewhere are not like this. In rock under load, microcracking is initially distributed through the volume and then **localizes into a narrow band which becomes the eventual fracture surface**, so that activity in that band rises steeply while activity elsewhere declines. The accumulated damage of the specimen as a whole is increasing throughout. **A sensor placed outside the band therefore records a decrease during precisely the interval in which the system is closest to failing**. The same structure appears wherever a process concentrates: a measurement confined to one region of a plasma, one part of a network, or one component of a loaded structure reports the behaviour of that region and not of the system.

The consequence for membership, in both directions. A candidate system should not be excluded because a measured signal decreases, if that signal reports a part of an extensive quantity rather than the quantity itself; the decrease may be the complement of an increase elsewhere. **Nor should a candidate be accepted because a local signal rises**, since a localizing process produces exactly that in the region where it concentrates, whatever the rest of the system is doing. The condition stated in Section 4.2 applies to the whole, and establishing that a measurement reports the whole is part of the work of identifying the state variable.

What makes this harder than ordinary sampling error. A noisy sample can be improved by averaging, and the error it introduces is unbiased. **Anticorrelation is neither**: averaging a fixed region for longer does not remove it, because the region is genuinely losing what the rest is gaining, and the bias has a direction set by where the process concentrates. **The worst case is also the most interesting one**, since localization sharpens as the system approaches its limit, so the discrepancy between part and whole is largest exactly where the measurement matters most. Where the extent of the system can be covered, covering it is the remedy; where it cannot, the relation between the sampled region and the whole has to be argued rather than assumed.

4.4 Conjectures

Conjecture 1: the class is not empty. That some real system, in some domain, satisfies the conditions below, and therefore exhibits the behavior proved in Section 3. **A single verified instance would settle the existence question completely**, and no accumulation of unverified resemblances would do so. This asymmetry governs how the rest of this paper is written: candidate systems are examined one at a time, against explicit conditions, and a candidate that fails is not evidence against the conjecture, only a candidate that failed.

Conjecture 2: the fluctuations are generated by nesting. A further element distinguishes this construction from a system merely driven by external noise. The fluctuations are not assumed to be imposed from outside: **they may be produced by smaller, faster instances of the same equation nested within the system**, each running to its own singularity, being cut off by the system's own regularization, and contributing a

bounded excursion to the exponent of the slower level that contains them. In this picture, noise is not a modeling convenience added to a deterministic law, but the aggregate trace of the same dynamic operating at a finer scale. **The mathematical statement does not depend on any particular device for aggregating these contributions;** the simulations of Sections 4.3 and 5.1 implement the aggregation in a simplified form, and nothing in the argument rests on that choice. **What those simulations settle is that nesting suffices:** a cascade in which only the fastest level is driven externally, and every level beneath it is driven by the completions above it, reaches the singularity in the same way as one driven throughout.

The arrangement also has a standard description, which is some evidence that it is a construction rather than a contrivance. It is a piecewise-deterministic Markov process in the sense of Davis (1984): deterministic flow between events and jumps at events. With levels indexed from slowest to fastest and rates $k_1 < \dots < k_N$, the flow between events is

$$\frac{dy_i}{dx} = k_i (1 + y_i^{m_i})$$

and a *boundary jump* occurs when some y_i reaches its cutoff: that level and those above it are reset to the value beneath, $y_j \leftarrow y_{i-1}$ for $j \geq i$, and the exponent of level $i-1$ is raised for a bounded interval. **The jump is state-triggered, which is what makes the fluctuation internal:** nothing is drawn from outside at that step. **The ordering of the levels is then a property of the process and not an assumption,** since the region $y_1 \leq \dots \leq y_N$ is forward-invariant: the reset places the upper levels on the one beneath, and where two coincide the faster separates upward.

What remains open is whether nesting is necessary, and whether any real system is organized this way. Neither follows from the above, which shows only that the arrangement is consistent, self-contained once started, and expressible in a standard form.

Conjecture 3: the levels are coupled through a shared carrier. Conjecture 2 states where the fluctuations come from, but not how they reach the level they act on. **The coupling is presumably mediated by a single dynamic quantity that the levels share:** in a plasma, a quantity such as the local density, the current or the magnetic field, where the excursions to be understood are the smallest eddies and their abrupt detachment; in a financial system, the volume and velocity of capital flows; in the circuit of Section 6, the charge transferred per cycle through the coupling winding. **The carrier need not be a physical quantity.** It may equally be an informational one, as in the spread of an item over a network, where the carrier is the rate at which it is passed on: that rate rises while the item is being shared and falls while it is only being consumed. Such systems are finite and saturate, so at most their early dynamics could have this structure.

The circuit is a case in which the carrier can be specified rather than guessed, and specifying it shows that naming the quantity is not the whole question. There the charge delivered into the accumulator closes the gap, and the law by which it does so fixes the exponent; the rate is set by a combination of that same charging, the accumulator's capacitance and the slope of the oscillator ramp, as the coefficients derived in Section 6.3 show. **The same quantity would therefore act on the exponent or on the rate according to which of the two it is delivered into,** and by Proposition 5 only the first can produce a singularity where none would otherwise occur. In the other examples above neither the carrier nor its point of entry is settled, and both have to be established before a coupling can be said to exist at all.

Both cases of the circuit make this explicit. Writing C for the accumulator's capacitance and v for the slope of the oscillator ramp in volts per second, which the current sources hold constant throughout, the two charging laws give

$$\frac{df}{dt} = \frac{q}{C v} f^3 \qquad \frac{df}{dt} = \frac{I}{C v} f^2$$

with q the charge delivered per cycle and I the current delivered per unit time. **The exponent differs between them and the coefficient does not:** the same three quantities set the rate in both cases, while what distinguishes 3 from 2 is only whether the delivery is counted per event or per unit time. **A coupling can therefore be identified by what it changes:** one that alters any of q , C or v moves the rate, and one that alters how the delivery is counted moves the exponent.

The ramp itself does not change during a run. Its slope is fixed by the current sources, so the oscillator always traverses a given voltage difference in the same time. **The frequency rises because the distance to be traversed shrinks, not because the traversal becomes faster,** and an observer watching the waveform sees the same slope throughout while the period shortens.

If this is correct, every system in the class must possess such a carrier: a single quantity in which the activity of the finer levels is present and, at least in principle, accessible to measurement. The moving-average construction of Part I, Section 7.6, where the fluctuating exponent is driven by an average taken over the observed signal itself, is one concrete realization of this idea rather than an additional assumption.

Conjecture 4: the boundary of a finite system acts on the detachment, not on the equation. The systems this framework might apply to differ in how far their limits lie from the range in which they are observed. Some, such as a population in a closed habitat, are plainly finite; others appear unbounded simply because their limits are far away. **The conjecture is that this difference does not reach the equation.** In both cases the exponent presumably fluctuates around the critical boundary, and in both cases the instances run to their singularities. What finiteness changes is

what happens at the end of that run: **the closer a system is to its own limits, the sooner the observable must detach from the instance it was following**, because the physical budget that would be needed to continue no longer exists. A saturating growth curve would then not be evidence against the structure; it would be the detachment made visible at the level where measurement takes place (see *Regularized singularities* above).

This has a consequence that the unbounded case does not have. In a finite system the supporting structures are themselves finite in number. If the outer, slower instances run to their singularities one after another, with nothing left to replace them, **the observable eventually detaches from the last support that remains**, and the system collapses to whatever level lies beneath, which may be far below the one it had reached. On this reading, a collapse of that kind is not a separate phenomenon requiring its own explanation, and need not be preceded by any visible external cause: **it is the terminal case of the same dynamic**, reached because the hierarchy that carried the growth was exhausted rather than because the growth was interrupted. The reading also carries a prediction: **such a collapse should be preceded by the scaling behavior of Section 2 and by detachments that become more frequent at the outer levels**, and not by a gradual flattening. Whether the exponent of a finite system does in fact fluctuate around the boundary, rather than falling away from it as the limits are approached, is the part of this conjecture most open to test.

Section 5.2 gives this a quantitative form, which the conjecture lacked. The supply a closing gap demands scales as G^{2-m} , so for $m > 2$ it diverges as the gap closes, for $m < 2$ it falls away, and only at $m = 2$ is it constant. **A finite system is one whose supply has a ceiling**, and the detachment this conjecture describes is what happens when the demand reaches it.

This makes the timing of the detachment calculable rather than merely expected. Where $m > 2$ the demand grows without bound, so a ceiling of any height is reached, and the only question is when: the detachment is then not contingent on the system being close to its limits but is forced for every finite system, the nearer limit merely arriving sooner. **Where $m < 2$ no such forcing occurs**, since the demand is falling, and a detachment there requires some other bound: the observable saturating, the measurement losing resolution, or the gap reaching the scale of its own contents as in Section 4.1.

The conjecture is therefore better stated with a condition attached. For exponents above 2 the boundary of a finite system acts on the detachment and not on the equation, and it does so necessarily. **For exponents at or below 2 the claim is weaker**, since a bounded driver can follow such an approach to its end and the detachment, if it occurs, has a different cause. **What remains conjectural in both cases is that the equation itself is untouched**, which no part of this argument establishes.

5. BOUNDED SYSTEMS AND SIMULATIONS

A bound can enter in four distinct places, and they are independent of one another. It may cap the **observed quantity** while the dynamic beneath continues, as when an instrument saturates or a thermal capacity smooths; it may cap **each instance**, as the cutoffs of Section 4.3 do, in which case what matters is not the bound but what the levels above reset to; it may cap **the structure**, by making the number of levels finite, so that when the outermost instance completes there is nothing left beneath the observation; or it may cap **the coupling**, since in a bounded system the observable cannot detach from the slower levels without limit. **Which of the four governs is a question about the system rather than about the construction**, and a single bounded simulation would be a model of whichever choice its author happened to make.

Three scaling laws follow from Sections 2 and 3 without further assumption, and they are the usable part of this section. Each is stated conditionally: if a system is of this kind, and if its observed quantity is a superposition of the events of the level beneath, then these relations hold.

Event rate. An instance starting from the support level Y and running at rate k_f completes, by Proposition 3, after an interval $Y^{1-m}/[k_f(m-1)]$. The frequency of such completions is the reciprocal:

$$v(Y) \propto k_f(m-1) Y^{m-1} \quad (v \propto k_f Y \text{ for } m = 2)$$

The events grow denser as the support rises, as a power of the support with the feedback exponent in it, and their individual durations shrink in the same proportion.

Smoothing. A reading averaged over a window W contains $N = vW$ such events, and the relative fluctuation of a superposition falls as $N^{-1/2}$:

$$CV(W) \propto (vW)^{-1/2} \propto Y^{-(m-1)/2} W^{-1/2}$$

A long reading therefore smooths as a power of the support, at the same time and for the same reason as the events beneath it grow denser. This holds under the usual superposition assumptions, independent events of comparable shape, which the construction satisfies only approximately.

Crossover. Dividing the two gives the window length at which a reading stops resolving individual events and begins averaging them:

$$W^*(Y) \approx 1/v \propto Y^{-(m-1)}$$

which contracts as the support rises, and this is the quantity that actually moves. Nothing becomes simultaneously calmer and busier: **the resolving scale slides downward past whatever fixed window an observer happens to use**, so a long window crosses into the averaging regime while a short one remains in the resolving regime, and the same record presents both appearances at once. **Genuine stabilization, by contrast, reduces activity on every timescale.**

What follows for measurement is left open deliberately. The laws above say that something detectable happens as a bound is approached, and that it happens at different window lengths in different directions; they do not prescribe how to detect it. Comparing the variability of one record at two window lengths is one possibility, as is counting events directly, or comparing two measured quantities of the same system whose physical smoothing differs. **Which is appropriate depends on what a given field can measure and on what each of its channels already smooths**, and a reader in a position to apply these relations is in a better position than this paper to judge the limits of their own instruments. **What the construction supplies is the pair of scaling laws and the contracting crossover between them.**

A consequence of the first law deserves separate statement, because it determines where in a record the structure can be seen at all. The interval between successive events at a level is the reciprocal of the rate, so by Proposition 3

$$\Delta(Y) = 1/v(Y) \propto Y^{1-m} / k_f$$

which falls as a power of the support. The intervals are long and the events sparse while the support is low, and they contract sharply once it is high; for $m = 2$ the interval is inversely proportional to the support, so a thousandfold rise in the support compresses the intervals by the same factor. **Almost all of the events in a run occur in the last part of it.**

Now suppose the rate itself varies, as it will in any realization. Writing k_f as a fluctuating quantity, the intervals acquire a spread whose relative size is that of k_f and does not depend on the support. **The mean contracts along the trajectory and the relative scatter does not**, so the ratio of signal to scatter is unchanged, while the number of events available to measure it grows only at the end. **Early in a run the contraction is buried in the variation of the rate, and only late does it separate from it.**

This has a direct consequence for how such a record should be read, and it is not optimistic. A system of this kind, examined over its slow phase, presents intervals that are long, few, and widely scattered, with a trend too weak to extract. **Fitted over that phase it will resemble a memoryless process**, in which the waiting time to the next event is independent of how long it has been since the last; and a test for such independence, applied there, will not reject it. **A finding of memorylessness in the early record is therefore consistent with this construction rather than evidence against it**, and the distinction can only be drawn where the events are numerous, which is to say near the end.

The requirement this imposes is awkward but precise. To test the contraction one needs many events from the same system, taken from the part of its history where they are dense, and a record that reaches back far enough for the comparison to mean anything. **Where a catalogue instead aggregates events from many systems at many stages**, the contraction of each is averaged against the others and the result is the distribution the aggregate happens to have, which carries no information about any one of them.

Two further things a bounded system does that the idealized one does not. First, **detachment of the reading is not confined to the neighbourhood of a true asymptote**; it may occur at any earlier stage at which the reading can no longer follow. Second, where the levels do not share a noise source, their trajectories need not stay ordered: **one instance may flatten while a slower one beneath it completes**, so the observed quantity can enter a prolonged plateau that terminates in an event not visible in it at all.

5.1 Plateaus

A system of this kind need not look like one. The simulation below produces **a level that flattens into a long plateau and nevertheless terminates in a singularity**, and it is included for a methodological reason: **a candidate system that fails to show the expected shape is not thereby excluded**, and the shapes this construction can produce are more varied than the idealized figure of Section 4.3 suggests.

What is varied. Three levels run, with rates in the ratio 30 : 70 : 100, and the middle level begins part-way through, taking its starting value from the level beneath. **Only the fastest level draws on an external noise source**; the other two receive their fluctuations from the level above them. Each time the fast level completes, it raises the exponent of the middle level from its subcritical base value to a supercritical one for a fixed number of steps, and each time the middle level restarts, it does the same to the level beneath. **The damping that produces the plateau acts on the duration of those excursions rather than on their size**: as the middle level's observed quantity rises past a threshold, the number of steps for which it is held supercritical is divided by that quantity, so the boosts become shorter as the level grows. **This is the one element imposed by hand**, and it stands for a coupling that weakens at high frequency rather than for anything derived.

Both coupled levels sit below the critical boundary except when an excursion arrives. Their base exponent is 0.85, and each boost raises it to 1.1 for the middle level and to 1.3 for the level beneath, in both cases for a limited number of steps. **The mean therefore lies below the boundary throughout**, and by the theorem of Section 3 this is no obstacle, since what is required is a positive probability of sustained excursions above the boundary and not a favourable mean. **Shortening those excursions changes when the singularity arrives, not whether it does**, and what the

shortening buys in the record is the plateau: a long, almost flat stretch in place of a visible acceleration. The flattening and the eventual termination are the same mechanism seen at two amplitudes.

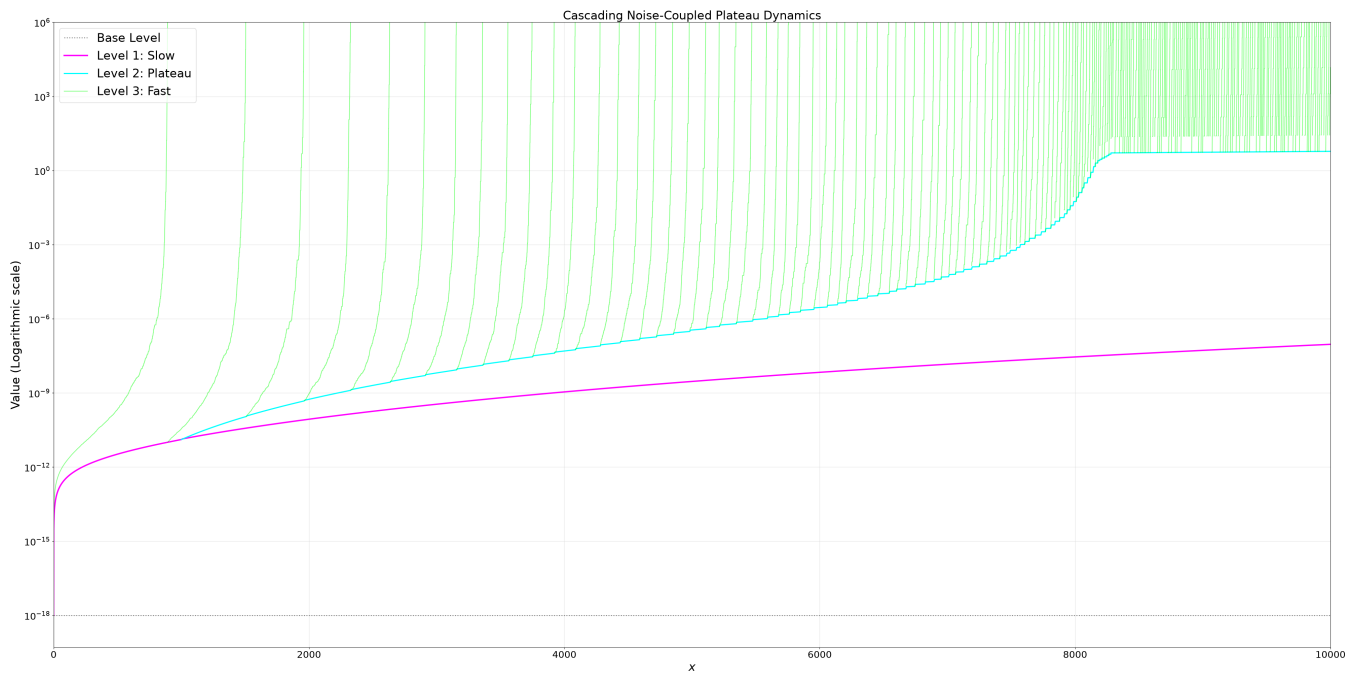


Figure 4.

Four features of the figure are worth reading carefully.

1. The plateau is quiet on its own channel and loud on the one above it. When the middle level establishes itself, the level running on top of it begins to terminate far more often, and the change is abrupt. This is the event-rate law of Section 5 at work: the fast level now rests on a support many orders of magnitude higher, and its completion interval shortens accordingly. **An observer watching the plateau sees calm; an observer watching the faster level sees the opposite,** from the same system at the same moment.

2. The order of completion is not guaranteed. Where levels do not share a noise source, nothing orders their terminations, so a slower level may end while a faster one is still flat. For an observer reading only the flat channel, the event arrives from a structure not visible in that channel at all. **This is a difficulty for the observer and not for the system,** which continues or reassembles as described below; what the channel cannot show, it cannot show either before or after.

3. The plateau has an interior. Its apparent smoothness is a property of the level itself, not of what is happening within it, and the density of events above it is highest precisely where it looks flattest. **Smoothness in one channel is not evidence of quiescence in the system,** which is the same conclusion reached in Section 5 by a different route.

What this does not establish, and the risk it carries. The damping is imposed rather than derived, and it is the one piece of the arrangement chosen to produce the result it produces. The coupling itself is not: **the fluctuations of the two lower levels are generated by the completions of the level above them, which is Conjecture 2 of Section 4.4 realized rather than assumed,** and only the fastest level is driven externally. **The simulation therefore exhibits a mechanism and not only a shape,** with the qualification that the mechanism's one free element, how the coupling fades, was supplied rather than found. It remains an idealization in every other respect. The risk in a result of this kind should also be named: **a framework that accommodates both the sharp case and the flat one is in danger of accommodating anything.** What limits that danger is that the two are not interchangeable. The plateau version makes different predictions, and they are checkable: the event rate of faster structures should rise sharply as the plateau forms rather than falling with it; the termination should be delayed rather than removed; and the completion order of levels should not be reliable. **A plateau that is genuinely quiet at every scale is not this construction,** and that is the observation that would exclude it.

4. A plateau does not arise from the ordering of the levels, and it is worth seeing why not. Suppose a slower instance begins while a faster one is already running. Since the time to completion scales inversely with the rate, **the faster one finishes first whatever the order of their starts,** and the slower one is left beneath it as its support. The hierarchy therefore arranges itself: which instance carries which is decided by their rates and not by the sequence in which they appeared. **This produces a continuous hierarchy and never a plateau,** since at every moment some instance is running on whatever lies beneath it.

A plateau requires the coupling between two levels to weaken, not the levels themselves to change. The simulation above produces one by damping the exponent of the middle level by hand. In a physical realization the same thing happens for a reason, and the circuit of Section 6 indicates one. **That circuit carries a single instance, so the coupling in question is not yet present in it;** what it has is a winding transferring energy from the oscillator into the accumulator, which is the same channel that would carry a coupling between two such circuits if two were built

and joined. **A winding transfers energy only within a limited range of frequencies**, and the calculation below applies to either arrangement. For a fixed supply voltage the current in the primary rises linearly during the on-time, $I = V t_{on}/L$, so the energy delivered per cycle is

$$E_{cycle} = \frac{V^2 t_{on}^2}{2L}$$

which falls as the square of the on-time. As the frequency rises the period shrinks, and two choices present themselves, neither of which preserves the coupling. The on-time can be shortened to fit, in which case the energy per cycle falls quadratically and the transfer converges to zero even as the events become more frequent. Or the on-time can be held, in which case the core saturates: beyond saturation the inductance collapses, the current no longer stores energy, and the primary approaches a direct current through a resistance. **Either way the coupling fades out at high frequency, and the level beneath stops being fed.** What the simulation imposes by hand, the hardware would do for a reason that can be calculated in advance.

This places the plateau alongside the regularization of Section 4.1 rather than alongside the dynamics. There the limit was that a gap cannot close below the scale of its contents; here it is that a coupling cannot operate beyond the bandwidth of the channel carrying it. **Both are properties of the realization and not of the equation**, both can be located before a run rather than inferred from one, and both produce a record that looks like a change in the dynamic when nothing in the dynamic has changed.

A fading coupling is not a barrier to the singularity, and this is the point at which idealization misleads. The account above treats the transfer as smooth, so that it declines predictably toward zero. A real winding does not behave that way: **a single irregular pulse, arriving before the coupling has faded, can deliver enough energy in one collapse to close the remaining gap outright.** The plateau would then end not by a gradual resumption but by a single event. **What decides between the two is whether the transfer is smooth or irregular**, which is to say whether real noise is present, and the same conclusion has been reached from the mathematics in Section 3: the mean behaviour of a process determines very little, and what matters is the upper tail of what it occasionally delivers.

A plateau is a transition rather than a state, and the construction repairs itself. Suppose the coupling from a level fades, so that the level beneath it stops being fed and flattens. **The level beneath the flattened one does not stop:** its growth rate is bounded below by k , so it continues to rise while the plateau stands still. The two therefore converge, and in finite time the lower level reaches the stalled one. From that point the faster instances, which are driven externally, rest directly on it, and the supply of fluctuation is restored without the intermediate level participating at all. **The flattened level may simply cease to be part of the hierarchy.**

If an intermediate instance can form at all, one will form again. The probability of such an instance arising is either positive or zero; if it were zero the nested structure would not exist in the first place, so in any system of this kind it is positive, and the intermediate level reappears in due course. **The hierarchy is therefore not fixed by its history but reassembles from whatever levels are present**, which is the ordering property of Section 4.3 holding under loss as well as under growth.

This bears on the recurrence condition of Section 3. The scenario that would defeat it is one in which the probability of a successful window falls to zero, and a fading coupling looks like such a scenario. **The argument above suggests it is not a stable one:** the level beneath catches up, the faster instances come to rest directly on it, and the supply of completions is restored without the intermediate level participating. **The simulation does not switch couplings dynamically and so cannot show this**, which is one of the clearer places where a reader modifying the code would learn something the figure does not contain.

None of this is established, and the simulation is not evidence for it. The argument assumes that the lower level continues to rise, which holds in the construction and need not hold in a bounded realization, and that an intermediate instance can reform, which is an assumption about a system rather than about the equation. **It is recorded because it identifies something a modified simulation could settle**, and because the alternative, a coupling that fades permanently, is the one case in which the theorem's reach over the nested construction would fail.

If no intermediate instance reforms, the question is left open rather than answered. A level whose supply has faded and whose hierarchy has not reassembled is driven by whatever base fluctuation its system provides, and Section 6.3 establishes only that some fluctuation is present in any realization, not that it reaches the exponent. **In the one case where the base fluctuation can be named, it does not:** the thermal noise of the circuit acts on the gap and leaves the exponent untouched. **Whether a base fluctuation anywhere reaches an exponent is the question of Section 7.1**, and the fate of an isolated level turns on it.

What can be said is narrower. Where the hierarchy reassembles, the recurrence condition of Section 3 is restored and the theorem applies as before. Where it does not, the level continues under its own base fluctuation, and whether that produces a termination depends on what the fluctuation acts on, which differs from one system to another. **The coupling therefore determines more than the timescale in this case**, contrary to what a fading coupling might suggest, and the plateau is a transition only where something is left to take over.

```
import numpy as np
import matplotlib.pyplot as plt
```

```

# --- 1. PARAMETERS ---
A = 10e-17
B = 10e-5

# Updated growth rates for the three layers
kBase = 30 * B      # Level 1 - Magenta (Slow)
kTop = 70 * B       # Level 2 - Cyan (Plateau)
kFast = 100 * B     # Level 3 - Lime (Fast)

yInit = 1e-6
numSteps = 10000

# Branching point for the Cyan layer
startStepTop = 1000

# Exponents and boost configurations
mBaseBase = 0.85      # Base exponent for Level 1 (Magenta)
mBaseBoost = 1.3       # Boosted exponent for Level 1
magentaBoostSteps = 10 # Duration of boost for Magenta when Cyan initiates

mTopBase = 0.85       # Base exponent for Level 2 (Cyan)
mTopBoost = 1.1       # Boosted exponent for Level 2
cyanBoostSteps = 5     # Initial duration of boost for Cyan

# Plateau damping threshold on observable y-axis
plateauThreshold = 1.0

# Stochastic noise configuration for Level 3 (Green) only
p = 0.45
fastUpper = 1.2
fastLower = 0.8

# --- 2. NOISE GENERATION (Level 3 / Fast only) ---
mFastNoise = np.random.choice([fastUpper, fastLower], size=numSteps, p=[p, 1 - p])

# --- 3. ARRAY INITIALIZATION ---
yBaseVis = np.full(numSteps, np.nan)
yTopVis = np.full(numSteps, np.nan)

yFast = np.full(numSteps, np.nan)
yFastVis = np.full(numSteps, np.nan)

yBaseVis[0] = yInit
yFast[0] = yInit
yFastVis[0] = yInit

baseActive = True
topActive = False
fastActive = True

# Timers for exponent boosting
cyanBoostTimer = 0
magentaBoostTimer = 0

# --- 4. SOLVING THE DIFFERENTIAL EQUATIONS ---
for i in range(1, numSteps):
    # Dynamic exponent evaluation for Level 1 (Magenta)
    if magentaBoostTimer > 0:
        mBase = mBaseBoost
        magentaBoostTimer -= 1
    else:
        mBase = mBaseBase

    # Dynamic exponent evaluation for Level 2 (Cyan)
    if cyanBoostTimer > 0:
        mTop = mTopBoost
        cyanBoostTimer -= 1
    else:
        mTop = mTopBase

    # Stochastic exponent for Level 3 (Green)
    mFast = mFastNoise[i - 1]

    # --- L1: Base function (Magenta) ---
    if baseActive:
        dyBase = kBase * (1.0 + yBaseVis[i - 1]**mBase)

```

```

tempBase = yBaseVis[i - 1] + dyBase

if tempBase > 1e40:
    baseActive = False
else:
    yBaseVis[i] = tempBase

# --- L2: Branching plateau function (Cyan) ---
if i == startStepTop and not np.isnan(yBaseVis[i]):
    topActive = True
    yTopVis[i] = yBaseVis[i]

# Cyan branching triggers a boost in Magenta
magentaBoostTimer = magentaBoostSteps

elif topActive and i > startStepTop:
    dyTop = kTop * (1.0 + yTopVis[i - 1]**mTop)
    tempTop = yTopVis[i - 1] + dyTop

    if tempTop > 1e30:
        topActive = False
    else:
        yTopVis[i] = tempTop

# --- L3: Fast nested function (Lime / Green) ---
if fastActive:
    dyFast = kFast * (1.0 + yFast[i - 1]**mFast)
    tempFast = yFast[i - 1] + dyFast

    if tempFast > 1e20:
        yFastVis[i] = np.nan

# Dynamic plateau damping: boost duration scales inversely with observable Cyan height
if topActive and not np.isnan(yTopVis[i - 1]):
    currentFTop = A * (yTopVis[i - 1] / B)
    if currentFTop >= plateauThreshold:
        effectiveCyanBoostSteps = int(cyanBoostSteps / currentFTop)
    else:
        effectiveCyanBoostSteps = cyanBoostSteps

cyanBoostTimer = max(0, effectiveCyanBoostSteps)

# Cascading fallback logic
if topActive and not np.isnan(yTopVis[i]):
    yFast[i] = yTopVis[i]
elif baseActive and not np.isnan(yBaseVis[i]):
    yFast[i] = yBaseVis[i]
else:
    fastActive = False
else:
    yFast[i] = tempFast
    yFastVis[i] = tempFast

if not baseActive and not topActive and not fastActive:
    break

# --- 5. OBSERVABLE QUANTITIES ---
fBaseVis = A * (yBaseVis / B)
fTopVis = A * (yTopVis / B)
fFastVis = A * (yFastVis / B)

absoluteBaseLevel = A * (yInit / B)

# --- 6. PLOTTING ---
plt.figure(figsize=(40, 20))

plt.axhline(absoluteBaseLevel, color='gray', linestyle=':', linewidth=2, label='Base Level')

plt.plot(fBaseVis, color='magenta', alpha=1, linewidth=3, label='Level 1: Slow')
plt.plot(fTopVis, color='cyan', alpha=1, linewidth=2.5, label='Level 2: Plateau')
plt.plot(fFastVis, color='lime', alpha=0.5, linewidth=1.5, label='Level 3: Fast')

plt.yscale('log')
plt.title('Cascading Noise-Coupled Plateau Dynamics', fontsize=25)
plt.xlabel('$x$', fontsize=25)
plt.ylabel('Value (Logarithmic scale)', fontsize=25)

```

```
plt.tick_params(axis='both', which='major', labelsize=20)
plt.legend(loc='upper left', fontsize=25)
plt.grid(True, alpha=0.3)
plt.xlim(0, numSteps)

# Limit y-axis upper bound for cleaner visualization (increase if simulating larger values)
plt.ylim(top=1e6)

plt.tight_layout()
plt.show()
```

JupyterLab code for Figure 4.

5.2 Which Exponents a Physical System Can Carry

Mathematically the exponent ranges from zero upward without restriction, and nothing in Sections 2 or 3 privileges any value. A physical realization is more constrained, and the constraint can be stated exactly. From the relation of Section 6.2, whatever the regime, the gap closes at a rate

$$|dG/dt| \propto G^{2-m}$$

and this is the quantity the apparatus has to supply. It is a charge per unit time in the circuit, a displacement rate between surfaces, a transfer rate in any realization; whatever it is, something must deliver it, and the exponent determines how that demand behaves as the gap closes.

The three regimes follow immediately, and $m = 2$ separates them. For $m < 2$ the exponent $2-m$ is positive and the required supply falls to zero as the gap closes: the approach is a soft landing, demanding less of the driver the further it goes. **At $m = 2$ the supply is constant**, which is the unique case in which a driver of fixed capacity suffices from beginning to end. **For $m > 2$ the supply diverges**, so the singularity in the observed quantity is accompanied by a singularity in whatever is driving it.

That last case deserves emphasis, since it is easy to adopt an exponent above 2 without noticing what it commits one to. At $m = 3$ a fixed quantity is delivered per event while the interval between events shrinks to zero, so the power required grows without bound. **No physical driver does this**, and the description must therefore fail before the singularity it predicts, by the truncation of Section 4.1 and for reasons belonging to the driver rather than to the equation. **An exponent above 2 in a physical system is a transient description**, correct over the range where the supply can keep up and not beyond it.

It also explains the drift described in Section 8, and makes $m = 2$ something like an attractor. As a per-event delivery approaches the limit of what its channel can provide, the delivery ceases to be per event and becomes per unit time, which is exactly the passage from 3 to 2.

Intermediate values between 2 and 3 occur during that passage and not as a regime: a system can be found there, but it is crossing rather than resting.

What a passive arrangement can reach is narrower still. Section 6.3 sets out the cases: a charging path whose rate is proportional to what remains closes the gap exponentially and gives $m = 1$; one whose rate does not decay gives 2; one metered per event gives 3. **Values between 1 and 2 require the supply to decay as a fractional power of the gap**, which no passive path does and which must be imposed by a control law. **Values below 1 require the same and do not terminate at all** in the deterministic equation, subject to Section 6.3 on what happens to them in practice.

The practical conclusion is a prior rather than a rule. Confronted with an unfamiliar system of this kind and no information about its driver, the exponents to expect are those a driver of bounded capacity can sustain, which is to say $m \leq 2$, with higher values appearing as descriptions of a stretch rather than of a system. **This is a statement about drivers and not about dynamics**, and a system whose supply is effectively unlimited over the range of interest is not bound by it.

The results of this section and of Sections 2, 3 and 6.3 can be collected into a single equation, and doing so makes their relations visible. Everything said so far about a closing gap is contained in

$$dG = -c G^{2-m} dt + \sigma dW, \quad \text{absorbing at } G = G_{\min}$$

where the drift is the supply law derived above, σ is the fluctuation in the gap, and $G_{\min}$ is the scale at which the gap ceases to be defined, by Section 4.1. **Three parameters, and each of the mechanisms treated in this paper is one of them.**

Take the deterministic case first. With $\sigma = 0$ and $G_{\min} = 0$, the time to closure is obtained by separation:

$$T = \int_0^{G_0} \frac{dG}{c G^{2-m}} = \frac{1}{c} \int_0^{G_0} G^{m-2} dG = \frac{G_0^{m-1}}{c(m-1)}$$

which is finite if and only if $m > 1$, the integral diverging at the lower limit otherwise. **This is the criterion of Section 2 arriving from the other end.** There the question was whether the observable reaches infinity in finite time; here it is whether the gap reaches zero in finite time, and the same exponent decides both, as it must, since the two are reciprocals.

Now restore the fluctuation. For $m > 1$ it changes little: the drift already carries the gap to zero, and the noise perturbs when rather than whether. **For $m \leq 1$ it changes everything.** The drift alone leaves G positive for all time, but a diffusion with a bounded drift reaches any level it approaches, so closure occurs regardless; **the waiting time is then set by the noise and not by the dynamics**, which is the mechanism of Section 6.3 and the reason a subcritical setting terminates in practice though not on paper.

The three parameters correspond exactly to the three places a fluctuation can enter, and the correspondence explains their different effects. A fluctuation in m changes the exponent of the drift, and so changes whether the integral above converges: it can create a closure where none would occur, which is the theorem of Section 3. **A fluctuation in c changes only the coefficient**, leaving the convergence of the integral untouched: it alters the timescale and nothing else, which is Proposition 5. **A fluctuation entering as σ is additive rather than multiplicative**, so it does not scale with G and can carry the gap across a boundary the drift would never reach. **The asymmetry between them is a property of where each sits in the equation**, not a separate finding about each.

The absorbing level does the remaining work. Setting $G_{min} > 0$ truncates the approach at the scale of whatever occupies the gap, so no divergence is attained; the terminal exponent and the scaling laws of Section 5 describe the stretch before that level rather than a limit reached at it. **Raising G_{min} brings the truncation forward and lowering it defers the truncation**, and since σ does not shrink with G , closure by noise occurs at a few multiples of σ rather than at G_{min} whenever σ is the larger of the two.

Nothing here is new, and that is the point of stating it. Each result was obtained separately and is proved where it appears; what the collected form adds is that they are one equation read at different parameter values, which is easier to carry than five separate findings. **One assumption is doing work and should be named:** σ is taken to be additive, independent of G . A fluctuation proportional to the gap would give a geometric process that approaches zero without reaching it, and the conclusions about $m \leq 1$ would not hold.

5.2.1 What the Supply Law Does Not Cover

The scaling above describes a drift and treats the transfer as smooth. Where the transfer is granular it is not wrong, but it is incomplete: the demand G^{2-m} is evaluated at a gap the system passes through continuously, and a system that transfers in quanta does not pass through every gap. **It lands at discrete values and overshoots the last one**, as Section 8.5 derives.

The consequence is that the limit identified here can arrive earlier than the smooth law suggests, and by an amount the law does not contain. A step that overshoots leaves a gap smaller than the one the step was formed for, and the demand at that gap is correspondingly higher; where the overshoot is large relative to what remains, the next transfer begins under conditions the previous one did not anticipate, and a sequence of such steps can carry the system past the supply limit faster than a continuous approach would. **Whether this matters depends on the ratio of the quantum to the remaining gap**, which is small for most of a run and of order one at the end.

The simulations do not show this. They hold the exponent at its set value through the terminal stretch, which is an arithmetic convenience rather than a property of anything, and they transfer in fixed amounts without representing what an overshoot does to the conditions of the next transfer. **The treatment that covers it is in Sections 8.5.1 to 8.5.3**, where the counts, the overshoot distribution and the exponent as an interval quantity are set out.

This is not pursued further here, and the reason is one of scope. The supply law is a statement about what a driver must deliver, which is a question about physical realizations rather than about the equation; **the isomorphism this paper asserts is between equations**, and the behaviour of a particular driver near its limit belongs to whichever field supplies it. **What the law gives is where to look**, namely at the point where the demand crosses what the source can provide, and the granular correction says that point is reached sooner rather than later.

6. A PHYSICAL REALIZATION

The simulations of the preceding sections are numerical. This section describes **an electronic circuit whose governing equation is the one studied here**, and in which the feedback exponent is not a single designed value but a quantity the arrangement can be made to take across its whole range, from well above the critical value to below it. It has not yet been built, and nothing below is a measurement.

A convention followed from here on. Every exponent quoted in this paper is either *derived*, meaning obtained from an idealized description of a mechanism, or *measured*, meaning estimated from a record. **These are not interchangeable**, and the distinction is stated wherever an exponent is given. Of the values appearing in Parts II and III, those of the circuit and the mechanical arrangements are derived; the exponent of accelerating

creep in Part III, Section 2 is measured; the exponent of nonlinear ballooning theory in Part III, Section 1 is derived, which is the subject of the caution in Section 1.1 there.

The arrangement. A capacitor is charged at a steady rate. When its voltage reaches an upper threshold it is discharged, and when it falls to a lower threshold the charging resumes, so the circuit oscillates. The upper threshold is fixed. **The lower threshold is not a fixed level but the voltage on a second, independent capacitor**, referred to here as the **accumulator**, since its voltage only rises and it is this accumulation that drives the system. Writing G for the gap between the two thresholds, the oscillation frequency of a circuit of this kind is inversely proportional to that gap,

$$f \propto 1 / G$$

so that **as the accumulator voltage rises the window narrows and the frequency increases**. A comparator pair sets the two thresholds; because two comparators acting on the same node can contend near a transition, a small microcontroller is the simpler way to arbitrate between them, and it also provides the timestamping needed to record the output.

The accumulator voltage never decreases, whichever way it is charged, which is the non-decreasing state variable required by Section 4 realized structurally rather than imposed. The gap therefore closes monotonically, and the frequency rises monotonically with it. **What remains open is the law by which it closes**, and that is what fixes the exponent: by the relation of Section 6.2, the exponent follows from how the gap approaches zero and from how the observable reads it, and nothing else about the circuit enters.

Why frequency is the right observable. It can be counted rather than measured, so its precision is set by a clock rather than by an amplitude reference, and the resulting series needs no calibration in the sense of Section 4.3. It is nevertheless an *aggregating* readout in the sense of that section, since a frequency is always taken over a gate interval, and changing that interval changes what is reported. **This makes the circuit a test of the measurement problem as well as of the dynamic**: the same run, counted over two different gate lengths, should display the window-dependence described in Section 5.

Where the singularity stops. The binding limit is the one identified in Section 4.1: **the gap cannot close below the scale of what defines it**, which here is the input offset voltage of the comparators together with their noise. Once the gap is comparable to that offset, the two thresholds are no longer distinguishable and which one is higher is no longer determined, so the quantity whose reciprocal is being measured ceases to exist rather than becoming small. **This acts well before the dynamic limits do**: the comparator propagation delay and the slew rate of the charging stage bound the frequency only when the period approaches nanoseconds, whereas the offset is reached while the period is still long. **The regularization of this circuit is therefore a resolution limit and not a speed limit**, which is convenient, since an offset can be measured in advance and trimmed, while a resolution limit that has been characterized is a known truncation rather than an unknown one. **The divergence is therefore truncated by the apparatus, which is the regularization of Section 4.1 occurring for identifiable and measurable reasons**. Leakage from the accumulator sets a second limit, since the accumulated voltage is held against the comparator's input current and the leakage of whatever charges it; where that leakage balances the charge delivered, the accumulator stops rising and the frequency stops with it. **This limit must be characterized before a run rather than inferred from one**, since an accumulator that has reached its leakage equilibrium produces a plateau that could be mistaken for the dynamic.

What it would and would not establish. The circuit is a constructed instance, so it cannot show that the class described in this paper is non-empty in nature; it was built to be a member. **What it can do is check the predictions against a system whose exponent is known from its own construction**, which no field measurement allows: whether the terminal scaling follows Proposition 2, whether the event rate follows the law of Section 5, and how much the exponent estimator is biased by noise and by the choice of gate length. It is, in that sense, a calibration of the instrument rather than evidence about the world.

1. Charging by a coupled winding: $m = 3$. A fixed quantity of charge is delivered once per cycle, so the gap closes per event rather than per unit time. Then $G \propto (t^* - t)^{1/2}$ and the frequency diverges as $(t^* - t)^{-1/2}$.

Why the exponent is not a setting. Suppose each cycle transfers a fixed quantity of charge to the accumulator. The rate at which the accumulator rises is then proportional to the frequency, and since the frequency is inversely proportional to the gap, the gap closes according to

$$\frac{dG}{dt} = -b / G \quad \rightarrow \quad G(t) = \sqrt{G_0^2 - 2bt}$$

The gap reaches zero in finite time, at $t^* = G_0^2/(2b)$. Rewriting this in terms of the observed quantity $y = f \propto 1/G$ gives

$$\frac{dy}{dt} \propto y^3 \quad y \propto (t^* - t)^{-1/2}$$

arises because a quantity accumulating in proportion to a frequency closes a window whose width sets that frequency, so the exponent follows from the arrangement rather than from a component value. **It is nevertheless a derived exponent and not a measured one**, and it holds under

two idealizations that the hardware will not satisfy exactly: that the charge delivered per cycle does not depend on the frequency, and that the events are frequent enough for the continuum description to apply. **What the circuit is expected to produce is an exponent near 3 early and drifting from it later**, and the size of that drift is one of the quantities the measurement is for.

2. Charging at a constant current: $m = 2$. A current source charges the accumulator at a fixed rate, so the gap closes linearly, $dG/dt = -I/C$, and the frequency diverges as $(t^* - t)^{-1}$. **This is the arrangement actually used**, and its closing law is exact rather than asymptotic. Charging through a resistor toward a level above the threshold gives the same exponent by a different route: there the gap obeys $dG/dt = -(\Delta + G)/RC$ with $\Delta > 0$ the amount by which the charging level exceeds the threshold. As $G \rightarrow 0$ the closing rate approaches the constant Δ/RC , so the closing becomes linear in the limit and the frequency diverges as $(t^* - t)^{-1}$. A constant-current source gives the same exponent and closes linearly throughout.

A second case, and why it matters. If instead the charge transferred per cycle scales with the period, so that the transfer per unit time is constant rather than the transfer per cycle, the gap closes linearly and the same derivation gives $m = 2$. **Two plausible coupling behaviours therefore give two different exponents, and a real winding sits between them**, since the shape of the transferred pulse depends on frequency and departs from either idealization as the oscillation speeds up. The effective exponent is thus expected to drift during a run, for a physical reason, and to lie between 2 and 3. This is the situation described in Part III, Section 1.1 for a derived exponent, with the difference that here it can be measured: the slope of the logarithm of the growth rate against the logarithm of the frequency returns it directly, and the derivation above says what range the answer should fall in.

3. Charging toward exactly the threshold: $m = 1$. At $\Delta = 0$ the gap obeys $dG/dt = -G/RC$ and decays exponentially, approaching zero without reaching it. The frequency grows exponentially and no singularity occurs. **This is the critical boundary, and in this circuit it is a single point in the parameter space** rather than a regime: it requires the charging level to coincide exactly with the upper threshold, and any departure from that coincidence, in either direction, leaves it. **Since the charging level is set by a reference rather than adjusted by hand, how closely it can be made to coincide is a question of resolution**, and Section 6.3 shows that this sets a limit on how near the boundary the circuit can be brought at all.

4. Charging toward a level below the threshold: saturation, and not a subcritical exponent. At $\Delta < 0$ the gap stops at a positive value and the frequency levels off. Expressed in the observed quantity this is

$$\frac{df}{dt} = A f - B f^2$$

the logistic equation: the same superlinear term with its sign reversed. It is easy to mistake this for a subcritical exponent, since neither produces a singularity and in neither does the frequency fall. They are nevertheless different equations: a subcritical exponent gives unbounded growth as a power of time, $y \propto x^{1/(1-m)}$, which never levels off, whereas this settles at A/B . **Setting the charging level just below the threshold does not approximate a subcritical exponent; it produces a very high saturation level.** The arrangement needed is simply a fixed reference below the upper threshold, with the accumulator charged toward it; a current source fed from the same reference behaves the same way. **The frequency then rises, levels off, and stays there**, which is a behaviour the circuit can hold indefinitely and which is easy to mistake for the subcritical case until the record is examined for whether it is still rising.

What the four cases have in common. The passive circuit reaches $m = 3$, $m = 2$, the boundary at $m = 1$, and saturation. It does not reach anything between 1 and 2, and it does not reach anything below 1. The reason is the relation of Section 6.2: with $m = 1 + 1/(p q)$ and both factors positive, **any gap that closes in finite time gives $m > 1$** . A subcritical exponent requires the gap to approach zero more slowly than exponentially, as a power of time, which means the closing rate itself decaying as a power. **No passive charging path does this**: a resistor closes the gap at a rate proportional to what remains of it, which is always exponential, and a winding or a current source closes it at a rate that does not decay at all.

5. A charging current that decays with the gap: any exponent, including below 1. The restriction above is a restriction on passive charging paths, not on the arrangement as a whole. Suppose the charging current is made to follow the gap itself,

$$I \propto G^{1 + 1/a} \quad \rightarrow \quad G \propto t^{-a} \quad \rightarrow \quad m = 1 - 1/a$$

so that $a = 10$ gives $m = 0.9$, $a = 2$ gives $m = 0.5$, and $a = 1$ gives $m = 0$. **The gap then approaches zero without reaching it, and the frequency grows without bound and without a singularity**, which is the subcritical regime of Section 2 and is distinguishable from saturation precisely because it does not level off. The required current is a function of the present gap rather than of time, so the arrangement is a feedback loop and not a schedule: the gap is already measured by the circuit, and the charging current is set from it.

This exponent is imposed rather than found, and the distinction matters. The exponents of cases 1 to 4 follow from the geometry of the charging path, and within the idealization in which they were derived they could not have been otherwise; the exponent of case 5 is whatever the feedback law is written to be. **All five are derived values, and none has yet been measured. Nothing in case 5 contradicts the conclusion above**, which concerns what a closing gap does when left to a passive mechanism. It does mean that the arrangement as a whole spans the full range of the construction, from well above the boundary to well below it, within one piece of hardware.

Why this matters for the theorem. Cases 1 to 4 are all at or above the critical value, so an arrangement restricted to them can test the deterministic results of Section 2, the scaling of Proposition 2 and the event-rate law of Section 5, but not the theorem of Section 3, whose subject is an exponent crossing the boundary. **With case 5 available the crossing becomes constructible:** a supercritical branch and a subcritical one, switched between at random, give an exponent process whose mean can be placed below 1 while its upper tail remains above. That is the configuration the theorem concerns, and it can be made to fluctuate at a chosen amplitude and persistence. **Whether the mean or the upper tail decides the outcome is then a measurement rather than a simulation.**

A caution that applies to case 5 alone. Because the law there is imposed, two artifacts belong to the apparatus rather than to the dynamics. The charging current is adjusted at discrete intervals, so the realized law is a stepwise approximation of the intended one, and the deviation grows as the gap closes and the intervals become long relative to the remaining time. And the gap is measured rather than known, so the feedback inherits the resolution limit of Section 4.1. **Both are quantifiable on this hardware precisely because the intended exponent is known,** which makes them a calibration of the estimator rather than a limitation of it.

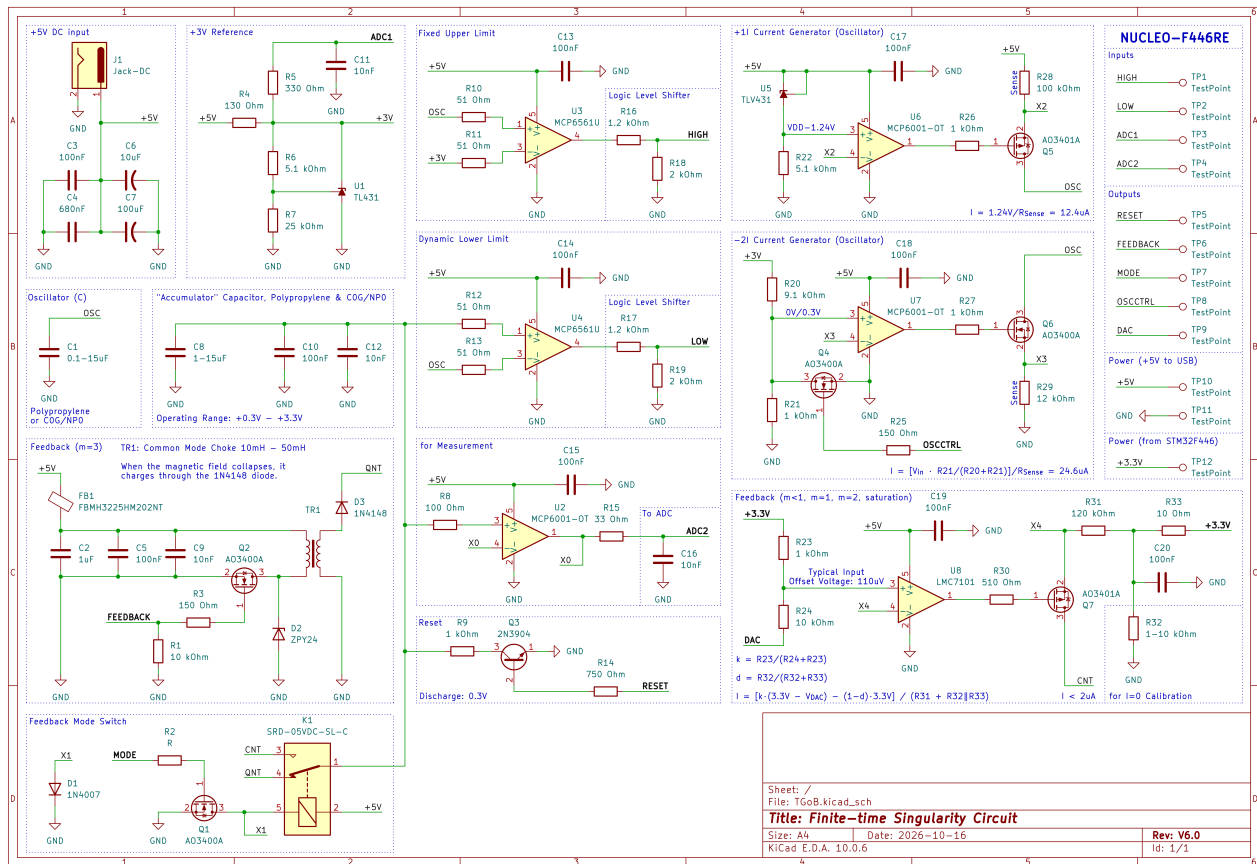


Figure 5.

6.1 Mechanical Analogues of the Same Equation

The circuit of the preceding section is not the simplest realization of its own dynamics. **A mechanical system of two surfaces and a single ball between them obeys the same equation**, in one dimension throughout, and setting the two side by side shows which features of the construction are essential and which belong to the particular hardware. The correspondence is offered as an analogy between equations; no claim is made that either system is a model of the other.

The elementary case. Let a ball travel at speed v between two parallel surfaces separated by a gap G , and let its speed be unchanged by each impact, so that no energy is exchanged. **This requires the surfaces to absorb the work they do**, by acting as a heat bath or by being held at rest at each impact and advanced between impacts; it is not what slowly moving elastic surfaces do, for the reason given below. The interval between impacts is G/v , so the collision frequency is

$$f = v / G$$

which is the relation $f \propto 1/G$ of Section 6, with the gap between two surfaces in place of the gap between two thresholds. **Now let the surfaces approach at a constant rate $u = -dG/dt$.** Differentiating,

$$\frac{df}{dt} = \frac{v}{G^2} \frac{du}{dt} = \frac{u}{v} f^2 \rightarrow f \propto (t^* - t)^{-1}$$

This is the equation of Section 2 with $m = 2$, and the terminal scaling $1/(m-1) = 1$ that Proposition 2 requires for that exponent. The singularity occurs at $t^* = G_0/u$, when the gap closes.

The two drives, and why the circuit gives a different exponent. The same arrangement yields a different exponent depending on what closes the gap. If the gap closes *per unit time*, as above, the result is $m = 2$. If instead it closes *per event*, by a fixed amount δ at each impact, then $dG/dt = -\delta f = -\delta v/G$, so G^2 decreases linearly and

$$G \propto (t^* - t)^{1/2} \quad f \propto (t^* - t)^{-1/2} \quad \frac{df}{dt} \propto f^3$$

which is $m = 3$, the exponent derived for the circuit of Section 6. **The difference between the two is not a difference of substrate but of bookkeeping:** in the circuit the coupled winding delivers its charge once per cycle, so the closing is per event; between two approaching surfaces the closing is per unit time. **The same structure therefore produces $m = 2$ or $m = 3$ according to whether the accumulating quantity is driven by the clock or by the events themselves.** This is worth noting beyond these two examples, since the distinction is easy to overlook in a candidate system and changes the predicted exponent by a whole unit.

What accumulates, and what does not. In the elementary case the ball's momentum is constant: it neither gains nor loses speed, and the frequency rises solely because the gap narrows. **The accumulating quantity is the displacement of the surfaces, not anything belonging to the ball.** The distinction matters, because the obvious alternative is to treat the ball's speed as the state variable, and that choice does not give a singularity at all.

Connection to Fermi acceleration. If the surfaces are *not* at rest at impact, each collision changes the ball's speed. In the classical treatment of this arrangement, the Fermi–Ulam model, randomly phased wall motion produces a diffusive growth of speed, with v increasing roughly as the square root of time. **Unbounded as that is, it is not a finite-time singularity**, and the reason is exactly the distinction of Proposition 5. Taking the surfaces to approach at rate u while also moving at impact, each collision adds $2u$ to the speed, and carrying this through the same derivation gives

$$\frac{df}{dt} = 3 \frac{u}{v} f^2$$

This relation is instantaneous rather than asymptotic, because the speed is no longer constant. Note that making the surfaces slow does not recover the previous case. Each impact adds $2u$, which vanishes as $u \rightarrow 0$, but the number of impacts grows as $1/u$, and the two cancel exactly: the invariant below holds whatever the closing speed. **The quasi-static limit of elastic surfaces is therefore the adiabatic case and not the isothermal one**, and the two differ by where the work goes rather than by how fast the surfaces move. Dividing the two rates gives $dv/v = -dG/G$, so vG is conserved: this is the adiabatic invariant of the one-dimensional billiard, and it holds increasingly well as the speed grows past the wall speed. With $v \propto 1/G$ the frequency scales as $f \propto 1/G^2$, and a gap closing at constant rate then gives

$$\frac{df}{dt} \propto f^{3/2} \quad f \propto (t^* - t)^{-2}$$

so that moving walls change the exponent from 2 to 3/2 rather than merely rescaling it. This is not a counterexample to Proposition 5, which concerns noise in the rate of an equation whose exponent is fixed; here the relation between the two state variables is itself altered, which is a change of structure rather than of coefficient. **The distinction is worth keeping in view, since a mechanism can act on the exponent without being a fluctuation at all.**

One feature of the Fermi–Ulam literature bears on assumption (S3) of Section 3, though the parallel should not be pressed too far. **With sufficiently smooth, deterministic wall motion the energy a particle can reach is bounded**, because invariant tori in the phase space separate the low-speed region from the high-speed one and cannot be crossed; with randomly phased motion they are absent, and the speed grows diffusively without bound. **The dividing line is not the strength of the driving but whether successive impacts are effectively independent**, and that is the same quantity assumption (S3) concerns, arising in a system with no connection to the present construction. **Two differences matter.** The growth there is diffusive rather than a finite-time singularity, so the consequence of independence is not the same consequence; and the independence enters through the smoothness of the driving rather than through a renewal structure. **The resemblance is therefore suggestive rather than evidential**, and it is recorded because the same distinction turns up in both places, not because one supports the other.

Where the exponent itself can move. Neither of the above makes m fluctuate. What does is the collision law. If the restitution of the impact depends on impact speed, the loss per collision scales as some power of the speed, and that power enters the exponent directly. **For real materials this dependence is measured rather than assumed:** for viscoelastic bodies the coefficient of restitution decreases with impact speed, following a fractional power. The exponent is then a material property, in the same way that the exponent of a hyperabrupt varactor is a property of its doping profile rather than of the circuit around it. **A mechanical realization of a fluctuating exponent would therefore be built by varying the collision law, not the wall motion.** By the relation of Section 6.2 the exponent is fixed by how the gap closes and how the observable reads it, so

an intervention reaches the exponent only by altering one of those two. Wall motion alters neither: it changes how fast the gap closes, which is the rate. **A collision law that depends on impact speed alters the second**, since the relation between the gap and the observed frequency then changes with the state. This is the mechanical statement of the distinction this paper draws throughout.

A caution from the simplest case of all. A ball bouncing on a single fixed surface under gravity, losing a fixed fraction of its speed at each impact, undergoes infinitely many collisions in finite time and comes to rest at a definite moment. **Here the collision frequency diverges while the ball's speed goes to zero**, and the accumulating quantity is the energy already lost rather than anything that grows. This is the reverse of the case treated above, and it carries a warning for the measurement problem of Section 4.3: **a diverging event rate does not establish that any state variable is diverging**, and a record of event times alone cannot distinguish a quantity running away from one running down. Which of the two is occurring follows from the governing equation, not from the frequency.

What these analogues establish. Nothing about the physical world: each is a textbook arrangement, and the correspondences are between equations. **What they do show is that the exponent of this construction is not a free parameter** but is fixed by the geometry of the accumulation in three independent realizations, an electronic one and two mechanical ones, giving 2 or 3 according to the bookkeeping rather than according to the substance.

6.2 What the Exponent Encodes: Closing Gaps and the Relation $m = 1 + 1/p$

The distinction drawn above between fixed and moving surfaces is not a peculiarity of this construction. It coincides with a distinction long established in the kinetic theory of gases, and the coincidence is worth recording, since it shows that the exponent separates two processes that were already known to differ.

Consider the same ball, of mass M , and average the force it exerts on one surface over many impacts. Each impact reverses the momentum, delivering $2Mv$, and impacts on that surface occur at a rate $v/2G$. The mean force is therefore

$$F = 2Mv \cdot \frac{v}{2G} = \frac{Mv^2}{G} \quad \rightarrow \quad F G = Mv^2$$

which is **the one-dimensional ideal gas law**, with G in place of the volume and F in place of the pressure; assigning $\frac{1}{2}kT$ to the single degree of freedom gives $F G = kT$. **This derivation is the standard one of kinetic theory and is not offered as a new result**; what follows from it is the point of this note.

The two cases of this subsection are the two classical compressions. Where the surfaces are at rest at impact, the speed is constant, so $F \propto 1/G$. That is Boyle's law, and the process is **isothermal**; it is also the case that gives $m = 2$. Where the surfaces move at impact, the invariant derived above is $v G = \text{constant}$, so that

$$F = \frac{Mv^2}{G} \propto \frac{1}{G^3} \quad \rightarrow \quad F G^3 = \text{constant}$$

and $F G^\gamma = \text{constant}$ with $\gamma = 3$ is **the adiabatic law for a one-dimensional monatomic gas**, for which $\gamma = 1 + 2/d = 3$ at $d = 1$. **The restriction to one dimension is not incidental and applies to every derivation in this section**: in three dimensions the same compression gives $\gamma = 5/3$, and the corresponding exponent is $m = 7/4$ rather than $3/2$, since the contents then gain speed more slowly as the gap narrows. That is the case giving $m = 3/2$. **The feedback exponent therefore distinguishes isothermal from adiabatic compression**: the walls that take no energy from the ball leave the temperature unchanged, and the walls that do work on it heat the gas.

Two things are worth taking from this, and neither is a claim about thermodynamics. First, **the exponent of this construction is not a free parameter even in the simplest mechanical case**: its two available values correspond to two processes that were distinguished long before this equation was written down, and the correspondence is exact rather than suggestive. Second, the quantity that accumulates differs between the two, being the displacement of the surfaces in the first case and the energy of the ball in the second. **Identifying which one accumulates is the same question that Section 4 requires a candidate system to answer**, and here it has a textbook answer, which is rarely the case elsewhere in this paper.

A rule covering every case derived so far. The exponents obtained in this section and the preceding one were each derived separately, but they follow from a single relation. Let the observed quantity diverge as a power of the remaining time,

$$y \propto (t^* - t)^{-s} \quad \rightarrow \quad m = 1 + 1/s$$

which follows by differentiating: $dy/dt \propto (t^* - t)^{-(s+1)} = y^{(s+1)/s}$. **The exponent is fixed by the terminal rate of divergence of whatever is observed, and by nothing else.** In the arrangements of this section that rate factors into two parts, because the observed quantity is a frequency and a frequency is a speed divided by a gap. Writing $G \propto (t^* - t)^p$ for the closing of the gap and $y \propto G^{-q}$ for the reading taken of it,

$$s = p q \rightarrow m = 1 + 1/(p q)$$

Here p belongs to the mechanism that closes the gap and q to what is being read. Where the speed is constant the frequency is simply the reciprocal of the gap, so $q = 1$ and the relation reduces to $m = 1 + 1/p$. Where the contents gain speed as the gap narrows, as they do whenever the surfaces do work at impact, $q > 1$. Applied to the cases treated here:

Arrangement	Gap closes as	p	q	m
Surfaces approaching at constant rate; accumulator charged at constant current	linearly	1	1	2
Fixed amount removed per event; accumulator charged by a winding	$\sqrt{t^* - t}$	1/2	1	3
Approaching surfaces that also move at impact, in one dimension (adiabatic case)	linearly	1	2	3/2
Inverse-square attraction (gravitational or electrostatic), idealized and taken as a separation closing	$(t^* - t)^{2/3}$	2/3	1	5/2

Only the behaviour near the singularity enters the table. A case not listed there illustrates the point. Charging an accumulator through a resistor toward a fixed level does not close the gap linearly, but exponentially, as $dG/dt = -(\Delta + G)/RC$ where Δ is the amount by which the charging level exceeds the threshold. **As $G \rightarrow 0$ the rate approaches the constant Δ/RC ,** so the closing becomes linear in the limit and the exponent is the same as for a constant-current source, which closes it linearly throughout. The two differ in the approach and agree at the end, which is all that m records. This also makes plain why $\Delta > 0$ matters: at $\Delta = 0$ the gap closes as $e^{-t/RC}$ and never reaches zero, which is the critical case $m = 1$, and at $\Delta < 0$ it stops at a positive value and the behaviour is self-limiting.

Two consequences, and the second is a limitation. The first is that an exponent measured from data is a statement about the closing law and can be read as one: where the possible closing mechanisms of a system are known and give different values of p , the measurement discriminates between them, which is the situation described for accelerating creep in Part III. **The second is that the exponent does not identify a mechanism on its own.** Two systems with no physics in common return the same m whenever their gaps close in the same way, as the first row of the table shows for an electronic and a mechanical arrangement. **Agreement of exponents across domains is therefore not evidence of a shared mechanism,** and the isomorphism this paper claims is an isomorphism of equations for exactly that reason. The membership conditions of Section 4 ask for the governing equation and the accumulating quantity precisely because the exponent alone does not supply them.

A third point follows for measurement. The relation above assumes the observed quantity is the reciprocal of the gap. **Read a different quantity and the same system returns a different exponent:** in the inverse-square case above, taking the speed rather than the reciprocal separation as the observable gives $m = 4$ for the same motion. This is the mapping of Section 2 appearing again, and it is the same point Section 4.3 makes about readouts: the relation between the record and the underlying state has to be settled before any exponent read from it means anything, since which quantity is being read changes the answer.

6.3 The Boundary Cannot Be Approached

There is a limit to how closely the boundary can be approached from below, and it is not a limit of the apparatus alone. Suppose the charging law of case 5 is written for an exponent just under the critical value, say $m = 1 - \epsilon$ with ϵ as small as the arithmetic allows. On paper the trajectory is subcritical and no singularity occurs. **In any physical realization the exponent is not realized exactly:** it is the outcome of components, measurements and a control loop, each with its own noise, so what the system actually runs is an exponent fluctuating about the intended value with some spread δ .

Where $\epsilon < \delta$, the system is not subcritical at all. Its mean lies below the boundary, its upper tail lies above it, and excursions across the boundary recur for as long as the system runs: that is precisely the configuration of the theorem of Section 3, and the singularity follows almost surely. **A setting intended to demonstrate subcritical behaviour would then demonstrate the theorem instead.** In an idealized circuit the relevant spread comes from the resolution of the control loop, the offsets of its amplifiers and the Johnson–Nyquist noise of its resistances, but nothing in the argument is electronic: any realization has some δ , set by whatever determines its exponent, and the quantity varies by many orders of magnitude between domains without ever vanishing.

Two consequences follow, and the second is the one worth keeping. First, **the subcritical regime is reachable but the boundary is not approachable:** there is a minimum distance below the critical value at which a system can be said to be subcritical at all, and it is set by the precision with which its exponent is realized rather than by the equation. Second, and generally: **a claim that some system sits just below the critical value is weaker than it sounds unless the precision of that claim is stated.** Without it, "just below" and "fluctuating across" are the same assertion.

It would be natural to try to measure δ on an ideal circuit, by moving the intended exponent toward the boundary from below until singularities begin to appear, and reading δ off the value of ϵ at which they do. **That measurement does not work, for a reason set out later in this section:** a subcritical setting terminates in any case, by a mechanism that does not involve the exponent at all, so the appearance of

singularities establishes nothing about δ . **The two mechanisms differ in the record that precedes the event rather than in whether the event occurs**, and any estimate of δ has to separate them first.

A subcritical setting is reachable but not sustainable, and the reason is the same noise that limits how closely the boundary can be approached. Consider the circuit with every component ideal except in one respect: the resistances carry Johnson–Nyquist noise, which follows from temperature rather than from imperfection. Set the charging law to a subcritical exponent, so that the gap closes as a power of time and the deterministic trajectory never arrives. **The gap nevertheless carries a fluctuation that does not shrink as it closes.** The deterministic part falls toward zero, the fluctuation does not, and at some point the sum crosses zero.

The crossing is irreversible. One threshold passes the other, the oscillator stops, and nothing in the arrangement restores the ordering; **a single excursion suffices, and no sustained one is required.** The accumulator need not even still be charging: a gap held near zero and then disconnected will be crossed eventually by the fluctuation alone. **The exponent is below the critical value throughout, and the event occurs regardless.**

This is a third route to termination, and it is neither of the two described so far. It is not the theorem of Section 3, which requires the exponent to cross the boundary and to stay across for a sustained interval; here the exponent never moves. It is not the metastable escape of Part III, Section 1, which requires a threshold in amplitude; here there is none. **It is a first-passage event:** a monotonically shrinking quantity with additive noise reaches zero because the noise does not shrink with it. **Nothing about this contradicts the mathematics**, since the deterministic equation describes a deterministic system and this one is not; what it shows is that the subcritical regime of a physical realization is not the subcritical regime of the equation.

The practical consequence corrects what was said above about case 5. A charging law written for an exponent below the critical value does produce subcritical growth, and the record will show it for as long as the run lasts; **it does not produce a system that will not terminate.** How long it lasts depends on the ratio between the gap and the fluctuation, which is calculable in advance, and the expected waiting time grows rapidly as that ratio grows. **A subcritical circuit is therefore one that terminates later, not one that does not terminate**, and the same is true of any physical realization driven by a closing gap.

It is distinguishable from the theorem, but only in the record before the event. Where a sustained excursion carries the system across, the approach accelerates: the frequency rises over many cycles in the manner of Proposition 2, and the terminal scaling is visible. Where a single fluctuation closes the gap, **there is no approach at all:** the record is flat to within its own noise and the event arrives in one cycle. The two therefore differ in what precedes them rather than in the fact of termination, which is the same conclusion reached for other pairs of mechanisms throughout this paper. **On this hardware the distinction can be checked**, since the intended exponent is known and both cases can be produced deliberately.

This names the base fluctuation in one case, which is part of what Section 7.1 asks for. Here it is thermal, its magnitude follows from temperature and the resistances involved, and it is present in the idealized circuit as much as in a real one. **What it does not do is answer the other part**, since this fluctuation acts on the gap and not on the exponent: it shows that a base fluctuation can be identified and calculated, not that one reaches an exponent anywhere. **The instance establishes that the first question has an answer in at least one system**, and leaves the second where it was.

The argument can be made quantitative, and the result is that the gap never has to reach zero. The voltage fluctuation on a capacitor in thermal equilibrium has variance

$$\sigma^2 = k_B T / C$$

which depends on the capacitance and the temperature alone, the resistance cancelling between the noise it generates and the bandwidth it sets; this is the standard $k_B T / C$ result for the voltage on a capacitor in thermal equilibrium. **The gap is therefore a Gaussian variable of width σ about its nominal value**, and the rate at which such a variable crosses a level G is, by Rice's formula,

$$\lambda(t) = f_0 \exp \left(- \frac{G(t)^2}{2\sigma^2} \right)$$

with f_0 a characteristic rate set by the bandwidth. **The hazard is negligible while the gap is many standard deviations wide and rises steeply once it is not**, so termination occurs when

$$G(t) \approx \sigma \sqrt{2 \ln(f_0 t)}$$

which is a few σ rather than zero, and only weakly dependent on how long the system has been running. With the subcritical law of case 5, where $G \propto t^{-a}$ and $m = 1 - 1/a$, the gap reaches that level at a finite time, and the waiting grows as $(G_0/\sigma)^{1/a}$. **A smaller exponent, meaning a larger a , postpones the event and does not prevent it.**

In this particular circuit the thermal limit is not the binding one, and the numbers show why. An accumulator of a few microfarads at room temperature gives σ of the order of tens of nanovolts, while the input offset of the comparators is of the order of a hundred microvolts: **the gap ceases to be defined, in the sense of Section 4.1, some three or four orders of magnitude before the thermal fluctuation could close it.** The mechanism described here is therefore real and calculable but masked, in this hardware, by a larger effect of the same kind. **What distinguishes it is that it cannot be trimmed away:** an offset can be measured and corrected, and $k_B T/C$ can only be **reduced by cooling the system** or enlarging the capacitor.

This is a third place a fluctuation can enter, and it is worth naming alongside the other two. The equation offers three: the exponent, the rate, and the state itself. **A fluctuation in the exponent can produce a singularity where none would occur,** which is the theorem of Section 3. **A fluctuation in the rate cannot,** which is Proposition 5, since the rate multiplies a term that vanishes with the state and cannot carry it anywhere it was not already going. **A fluctuation in the state is different from both,** because it is additive rather than multiplicative: it displaces the state directly, by an amount that does not shrink as the state does.

That is why it matters here and why it does not matter everywhere. It requires an absorbing boundary: a value the state can reach and not return from. In this circuit the boundary is the closing of the gap, which reverses the ordering of the two thresholds and stops the oscillator, and the crossing is irreversible for that mechanical reason rather than for any reason in the equation. **Where no such boundary exists, additive noise merely jitters the state and nothing follows from it,** which is why the same fluctuation in a quantity that grows away from zero does nothing at all.

The three should be kept apart rather than grouped. They differ in what they require, what they produce, and what a record of them looks like. **The first requires the exponent to cross a boundary and produces an accelerating approach; the third requires the state to reach one and produces no approach at all.** Both end in a termination, and the second ends in nothing. **A system may admit more than one of them,** and this circuit admits the first and the third by different routes, which is the reason it can be used to tell them apart.

LIMITATIONS

7.1 The Origin of the Base Fluctuation

In every candidate considered here, the coupling between levels is the tractable part and the noise at the bottom is not. How a completion at one level reaches the level beneath it is a question each field can answer with its own methods: in the circuit it is the charge delivered through a winding into the charging law; in a rock mass a crack opens and the redistributed stress opens others, until something damps the sequence or it runs out. **These are matters of mechanism, and the fields concerned are better placed to settle them than this paper is.**

What no field answers is where the fluctuation at the lowest level comes from. Every nested account terminates in an instance that is not driven by anything within the structure, and the simulation of Section 4.3 makes this explicit: only the fastest level draws on an external sequence, and everything else is coupling. **That external sequence is assumed, and it is assumed in every account in this paper.** The theorem of Section 3 takes the fluctuation as given; the circuit of Section 6 would take it from the thermal and control noise of its own components; Section 6.3 establishes only that some fluctuation must be present in any realization, not what it is.

Two questions follow, and the second is the one worth asking. The first is local: in a given system, what carries the base fluctuation, and does it reach the exponent or only the rate? That has been answered for the circuit and could be answered for the mechanical arrangements, and the answer is specific to each. **The second is whether there is anything general to say:** whether the base fluctuation of unrelated systems has any common character, or whether the question has as many answers as there are systems. **A framework asserting a mathematical isomorphism has no right to assume the first,** since the isomorphism is of equations and says nothing about what drives them; but if the same kind of answer turned up in an electronic, a mechanical and a geological case, that would be a result of a different order from anything in this paper.

Two forms such an answer might take are worth distinguishing, since they differ in what would have to be true. The first is that a single microscopic fluctuation underlies every case, the thermal one being its instance in the circuit, where Section 6.3 shows it to be present and Section 8 shows it to be masked by coarser effects. **That would make the question one of scale rather than of origin:** the same fluctuation everywhere, reaching the exponent only where the system is fine enough to resolve it. The second is that a cascade's base fluctuation is supplied by some other process overlapping it in scale, so that what is ambient to one is structural to the other. **That process need not itself be a cascade:** what the lowest level requires is a recurring source of events, and in the circuit of Section 6 the source is an ordinary oscillator rather than a structure of any kind. **A hand on a switch, closing the winding circuit at irregular intervals, would serve as well,** since what reaches the accumulator is a sequence of deliveries and not the character of whatever produced them. **Neither is advanced here.** The first requires a mechanism by which such a fluctuation reaches an exponent rather than a state variable, which the circuit case specifically fails to provide; the second requires an identified coupling between systems, meaning some quantity that actually passes from one to the other. **An isomorphism of equations does not supply it,** not because such a coupling would be inconsistent with one, but because the two are separate claims: two systems obeying the same equation need have nothing to do with each other. **Were a coupling found, the nested description would apply to the pair** exactly as it applies within one system, and the isomorphism would be unaffected. **They are set down because they are the shapes an answer could have, and because they differ in what would refute them.**

This is a direction for measurement and not for further derivation. Every exponent here was obtained by writing down an idealization and reading off its consequences, and Section 8 sets out what that is worth. **Answering the question above requires systems that differ physically as much as possible, each examined with the same two questions: what carries the base fluctuation, and where does it enter.** Nothing in the construction predicts the answer, and a negative result, that no common character exists, would be as informative as a positive one.

Two closing remarks, both negative. Raising the variance of a fluctuating exponent is not a second route to any of this: by Proposition 6 it accelerates the trajectory, and by Proposition 0 that moves where the singularity falls on the time axis rather than changing the shape of the approach, so variance and rate do the same work. And nothing here bears on why one interaction should be weaker than another, which is a question with its own literature and its own name; what the construction says is only that a difference in strength sits in the rate and leaves the exponent untouched, so two interactions of the same form and very different strength are not thereby in different classes. **That is a reason not to exclude a candidate, and not an explanation of anything.**

One question raised by this part is sharper than it was, and is left open. The membership conditions of Section 4.2 require the fluctuation to reach the exponent, and that requirement has been decisive repeatedly. **Section 6.3 shows that it is not the only route to a termination.** There a subcritical circuit terminates because the gap itself carries noise and crosses zero, with the exponent never moving. The fluctuation reached the state variable rather than the exponent, and the event occurred regardless.

The question is whether the requirement should still be imposed at the lowest level. Within the hierarchy it clearly must: a coupling that moves only the rate of the level beneath cannot produce a singularity there, which is Proposition 5 and which decided several cases in this paper. **At the bottom, where the base fluctuation acts directly on a state variable rather than through a level above, the case is less clear,** since a quantity closing toward zero with additive noise reaches zero whatever the exponent does. **Nothing else in this work changes if the requirement is relaxed there:** the theorem concerns the exponent and is unaffected, the couplings between levels are unaffected, and the conjectures are unaffected. What changes is which systems can be excluded at the first step, and the answer bears on how the base fluctuation should be characterized in the first place.

7.2 A Simplification in the Simulations

In the simulations of Sections 4.3 and 5.1 the external fluctuation enters at the fastest level and nowhere else. A closed model has no outside, so something must be supplied; the construction itself does not say where, and placing it at the top is a choice made for the code rather than a claim about any system. **Section 6.3 shows that a realization cannot have an exponent free of fluctuation at any level,** so the simulation is in that respect poorer than a physical system rather than richer.

Little turns on the choice. Once a level is running within a hierarchy, what drives it is the completions above it, and those are large and frequent compared with anything ambient; the external sequence matters at the top, where there is nothing above to supply them. **Attaching it elsewhere, or distributing it across levels, changes which instances are driven directly and leaves the cascade intact,** since the theorem takes the fluctuations as given and the couplings propagate them downward whatever their origin.

What the arrangement does not settle is how such a fluctuation enters a real system, which is the subject of Section 7.1 and is not addressed by any choice made in the code. **A reader modifying the simulation should treat the placement as a parameter,** and the figures as showing one setting of it.

In the circuit the base fluctuation can be named, and it survives idealization. Suppose every component is ideal in the usual sense: the current sources exact, the comparators free of offset, the capacitors free of leakage. **The thermal noise of the resistances remains,** since it follows from temperature rather than from imperfection, and it appears as a fluctuation in the accumulator voltage. The gap between the thresholds is therefore not a fixed quantity but one with a spread about its nominal value.

This fluctuation acts on the gap itself and not on the rate. The rate here is the slope of the oscillator ramp, fixed by the current sources, and nothing in the accumulator touches it; what the noise perturbs is the state variable whose closing produces the singularity. **By itself it changes neither the exponent nor the outcome,** since the gap closes on average as the charging law dictates. **What it sets is the resolution with which the gap is defined at all,** and as the gap closes the fluctuation does not shrink with it, so the relative spread grows until the gap is no longer a value but a distribution. That point is the truncation of Section 4.1, and here its location follows from theory rather than from observation: the magnitude of the thermal fluctuation is given by temperature and the resistances involved, so the scale at which the gap ceases to be defined can be calculated before the circuit is built. **It cannot be measured directly,** since by the time the gap has reached that scale there is no longer a gap to measure, and what the instrument reports there is its own resolution.

7.3 The Base Exponent Is a Choice

In the simulations of Sections 4.3 and 5.1 each level carries a base exponent of 0.85, and nothing in the construction requires that value. It was chosen to be subcritical but not trivially so, which makes a particular point clearly: a level already possessing superlinear feedback, though not enough of it to terminate on its own, is carried across the boundary by the completions above it. **That is one case and not the general one.**

The same applies to the excursion value of 1.3. It was chosen high enough to be visibly supercritical and low enough to keep the figures readable, and the construction requires only that excursions reach above the critical value, not that they reach any particular height. **Lowering it toward 1 lengthens the runs and leaves the behaviour intact;** raising it shortens them and makes the terminal rises abrupt enough to obscure the structure they came from. **Neither value is derived from anything,** and both are parameters in the sense this section intends. **The two-valued form is itself an idealization, and Section 8.5.3 says why:** an exponent raised by a transfer of finite duration follows a curve rather than switching between levels, so a process taking one of two values is an approximation to that curve. It is adequate where the transfers are brief compared with everything else and misleading where they are not. **The simulation has no representation of the transfer interval at all,** raising the exponent for a fixed number of steps and dropping it again, which is the coarsest version of that approximation.

The base exponent can be taken to zero, and the cascade is unaffected. At $m = 0$ the equation reduces to $dy/dx = 2k$, a level rising linearly and possessing no feedback at all; the couplings still raise the exponent during their excursions, the completions still occur, and the hierarchy still orders itself. **What changes is where the superlinearity comes from.** With a subcritical base it is partly intrinsic and partly supplied; with a base of zero it is supplied entirely, and the levels contribute nothing to it but their arithmetic.

The second case is the cleaner one and the stronger claim. It shows that the construction does not require its levels to be predisposed toward anything: a cascade of linearly growing quantities, coupled so that each one's completions act on the exponent of the next, produces the same behaviour as one whose levels were already superlinear. **This bears on Conjecture 2,** which holds that the fluctuations need not be imposed; the same arrangement shows that the feedback need not be assumed either.

It also sharpens what Section 3.1 says about measurement. There a deep subcritical stretch was shown to flatten a record toward a power law; with a base exponent of zero the levels are flat by construction between excursions, and what a record of such a system shows is linear growth interrupted by events. **An observer fitting the quiet phase would find no feedback whatsoever,** not merely a weakened one, and would be right about the quiet phase and wrong about the system. **The value 0.85 is therefore the conservative choice for display and not the natural one,** and a reader varying the parameters should take the base exponent down as well as the noise.

The figure below is such a run, with every base exponent set to zero. The two slowest levels are straight lines on this scale and remain so throughout the window shown; their singularities fall outside it. **Nothing visible in them indicates feedback,** and the only sign that anything is happening is in the levels above, whose completions are doing all of the work.

The moving average in the figure is no exception to Section 4.3. It carries a visible cyclic trace, and it is tempting to read that as the accumulation showing through; it is not. The trace follows the cycles of the level above, at a level of its own and without reproducing their falls, and a longer window would attach it in the same way to the level beneath instead. **What it reports is the window, not the structure.**

A reader who wants to see how easily this misleads can arrange it deliberately. Replacing the series fed to the moving average with a slower level, and lengthening the window in proportion, produces a figure that looks much like the original one at a different scale; repeating this down the hierarchy produces a sequence of such figures. **The impression is of a self-similar structure,** and an irregular one, since no conventional fractal description fits it well.

The impression is manufactured. What was matched at each step was the window to the level, and the resemblance follows from Proposition 0: the levels run the same equation at different rates, so their trajectories have the same shape and differ only in their clocks. **Nothing self-similar was found, because the resemblance was supplied by the choice of window rather than discovered in the data,** and the levels differ from one another in ways a self-similar description would not permit. This is the argument of Section 9 arrived at by experiment rather than by reasoning, and it is worth performing once for that reason.

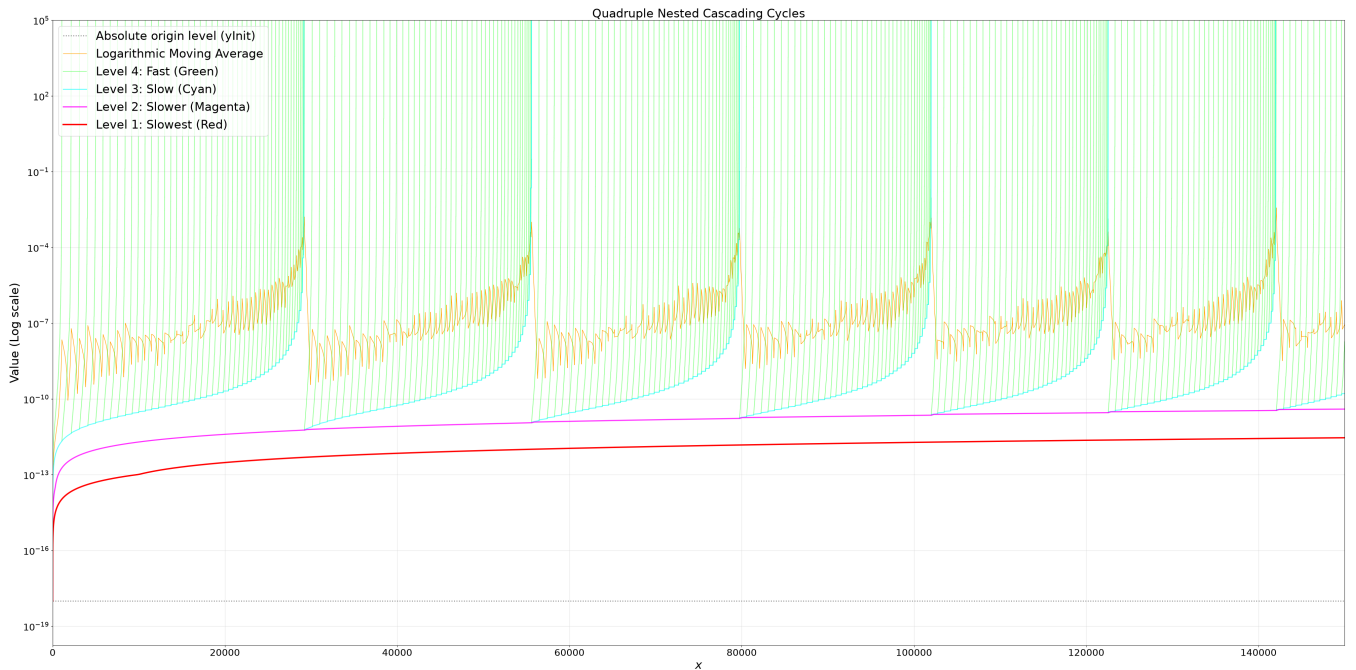


Figure 3b.

7.4 The Exposition Can Be Wrong Where the Objects Are Not

This paper contains two kinds of thing, and they do not carry the same weight. There are objects that can be checked independently of anything said about them: the equation and its closed solutions, the propositions and their proofs, the simulation code, the circuit and the measurements it would produce. **And there is the prose, which says what they mean.** The first kind is where the content is; the second is an account of it, written by someone who has been working on it long enough to have formed habits of reading it.

An account of a construction this size will contain errors that the construction does not. Several have already been found and corrected in the course of writing: a claim that a coupling acted on the rate when it acted on the state, a membership condition that would have excluded every candidate including the author's own simulations, a derived exponent treated as a property of hardware. **Each was a mistake in the reading and not in the thing read,** and each was found by going back to the object. There is no reason to suppose the remaining ones have all been caught.

The practical advice is to treat the objects as primary. The simulations are given in full and are meant to be run with other parameters; the circuit is specified well enough to build; the propositions are short enough to verify. **Where this paper's prose and one of these disagree, the object is right,** and a reader who finds such a disagreement has found something worth reporting rather than a difficulty of their own. **The conclusions drawn here are one reading of the construction and not the only available one,** which is also why the parameters are exposed rather than fixed.

This does not make the prose optional. What is proved is proved, and the distinctions it draws, between an exponent and a rate, between a state and an observation, between a derived value and a measured one, are the reason the objects can be interpreted at all. **The limitation is narrower: that the interpretation of a particular system, the assessment of a particular candidate, and the significance attached to a particular result are judgements,** and judgements of a kind this paper has already had to revise more than once.

8. CONCLUSIONS

8.1 What the Theorem Establishes

The result of this part is a theorem and three conjectures, and they are not equally secure. The theorem holds: where a feedback exponent fluctuates with a positive probability of sustained excursions above the critical value, and where successive excursions are effectively independent, a finite-time singularity follows almost surely whatever the mean exponent may be. **The theorem covers the nested case, and this was not obvious.** Its recurrence condition asks only that each window carry some fixed probability of a sustained excursion given everything before it, which independence supplies but does not exhaust. In the nested construction the windows are plainly dependent, and the condition holds anyway: the external sequence at the top is drawn afresh each time, and the supporting level never decreases, so by Proposition 3 a window that once sufficed always suffices thereafter. **The past can improve the prospects of the future and cannot worsen them,** which is what the conditional bound requires.

What the coupling between levels determines is the timescale, not the outcome. A fading coupling was the one case in which the theorem's reach over the nested construction appeared to fail: if each completion above delivers an excursion that shrinks faster than the requirement $\tau^*(Y)$ falls, the conditional bound is lost. **Section 5.1 shows that this does not survive examination.** Either an intermediate instance reforms and the hierarchy reassembles, or it does not and the lowest level is fed directly by the base fluctuation, which by Section 6.3 is never absent from a

physical realization. **Losing the intermediate levels makes the excursions rare and the waiting long, which by Proposition 0 is a change of clock and not of behaviour.**

The practical consequence is a shift in where the difficulty lies. The question is no longer whether a nested system reaches its singularity but whether it reaches it within any interval an observer could watch, and that depends on the depth of the hierarchy and on how much the finer levels deliver. **A system with its couplings intact arrives quickly; the same system with them gone arrives by the same route and may take arbitrarily long.** This is a limitation on observation rather than on the theorem, and it belongs with the measurement difficulties of Section 4.3 rather than with the conditions of Section 3.

It is worth being clear about what the theorem would be for if it holds. It states that a singularity occurs, not when, and the timing predictions of Section 3 are distributional rather than specific. **That makes it a framework for assessing whether a risk exists rather than a forecasting tool:** whether a slope's record shows recurrent excursions above the boundary is a question it can address, and when the slope will fail is not. Where a system is already understood and controlled, the use is different again and has nothing to do with prediction: it would say what has to be interrupted for the terminations to stop, which in the circuit of Section 6 is a matter of clamping the accumulator before the gap closes, and in a plasma would be a question for the field concerned. **A framework used for what it cannot do would be weakened by the gap above; used for what it can, much less so.**

The conjectures stand differently from one another. That the fluctuations can be generated by nesting rather than imposed is no longer open in one direction: the simulations of Sections 4.3 and 5.1 show that a cascade driven only at its fastest level behaves as one driven throughout, so nesting suffices. Whether it is necessary, and whether any real system is organized this way, is untouched. **That the class is not empty remains the central open question,** and the candidates of Part III leave it open: the plasma case belongs, on the established description, to a different class; the geomechanical case has the equation and a measured exponent but no identified irreversible state variable. **A single verified instance would settle it, and none is offered here.** Proposition 6 of Section 3.1 adds a bound that holds whatever the answer: a fluctuating exponent can make an early record look flatter than the deterministic case it matches, never steeper, so what feedback is visible early is at least present and more may be hidden.

8.2 Why a Measured Exponent Says Little

Measured exponents fare no better, and the simulation of Section 4.3 shows how badly. Every level in Figure 3 carries a base exponent below the critical value, and in every case a fit to the record returns a value above it. **The answer is known there because the code is known,** and it is wrong in the same direction for every level: the completions occur during excursions above the boundary, so that is what a fit can find, and nothing in the record points to the subcritical base beneath. **This is not a defect of the estimator but a property of what is available to be estimated,** and it applies to every candidate system in Part III, where no code is available for comparison.

A first source of drift arises in nested systems and is not an artifact at all. As Section 4.2.2 sets out, the rate an instance needs in order to complete within a given window falls as the support rises, so levels that contributed nothing early begin contributing later. **The closing of the gap then accelerates because the number of contributors grows,** which a fit to the terminal stretch reports as a rising exponent. Nothing in the dynamics has changed; the structure has gained levels. **This one differs from the other two in being a property of the system rather than of the description or the hardware,** and it is the only one of the three that a better instrument or a finer discretization would not remove.

An another source of drift is physical, and the two are not distinguishable from a record. Section 6 notes that a real winding couples differently at different frequencies, so the exponent of that arrangement moves between 2 and 3 for a reason belonging to the hardware. The discretization drift belongs to the description instead, and would be present even if the coupling were perfect. **Both produce the same signature,** and separating them requires an independent account of each rather than a better fit. **This is the sharpest statement this paper can make about measured exponents: a drifting exponent is the expected outcome, not an anomaly, and its direction and size carry information only once the sources of drift have been enumerated.**

8.3 Derived Exponents and the Limits of Idealization

Every realization in this paper is a continuum description of something that is not continuous. The ball crosses the gap a finite number of times; the circuit completes a finite number of cycles. The equation treats as a rate what is in fact a sequence of events, and that treatment is sound only while many events occur before the gap changes appreciably. **The approximation fails exactly where the exponent would be measured:** as the singularity approaches, the number of remaining events falls, and in the last part of the approach there may be only a handful.

That a discrete process is not fully captured by a continuous one is not a new observation, and the comparison of difference and differential descriptions is standard. **What is specific to this construction is where the discrepancy appears.** Here the observed quantity is itself an event rate, so the error does not sit beside the exponent as scatter: it enters the exponent. And since the number of events per unit of change falls monotonically along the trajectory, the error grows monotonically with it. **Discretization therefore shows up in a fit as a systematic drift of the exponent, in one direction, largest where the measurement matters most.**

The general lesson concerns derived exponents. Every exponent obtained in Part II was derived from an idealization: fixed charge per cycle, surfaces that take no energy at impact, a continuum of events. **Each is correct within its own construction and carries no warrant beyond it.**

The same holds for the exponent of nonlinear ballooning theory discussed in Part III, Section 1.1, and for the fixed exponent of the model from which this work began, which was later shown not to be recoverable from the data at all. **A derived exponent describes the description, and the only defence is to state, for every exponent quoted, whether it was derived or measured, and from what.**

Two consequences follow for what the circuit of Section 6 is for. It was presented as a realization of a known exponent, and that is now too strong: what it realizes is a mechanism whose exponent is near 3 early and departs from it later, by an amount that two identified effects predict. **That departure is a quantity to measure rather than an error to correct,** and because the intended exponent is known, the measurement calibrates the estimator used throughout this paper against a case where the answer is known in advance. **The second consequence is that the same caution applies to every candidate system in Part III,** where no such calibration is available and where the sources of drift are correspondingly harder to enumerate.

8.4 Where the Description Has No Content

A difficulty underlies the three just described, and unlike them it has no formulation here. The question that exposes it is simple to ask and has no answer in the present treatment: in the circuit of Section 6, when does the exponent change? The controller energizes the winding, the field builds, the transistor switches off, and the charge arrives at the accumulator over a further interval. **Each of these is a candidate moment and none of them is right,** since the continuous description assigns the system an exponent at every instant and the discrete process has nothing happening at almost every instant.

The difficulty is that the exponent is not a quantity the system possesses at a moment. What the circuit actually does is

$$G(t) = G_0 - (q/C) N(t)$$

with $N(t)$ a counter, so dG/dt is a sum of impulses: exactly zero between events and undefined at them. **The smooth law of Section 5.2 is not this derivative but a description of how often the events arrive,** and an arrival rate is meaningful only over a window containing many arrivals. **The exponent is a property of such a window and not of an instant,** and asking when it changed is asking which throw of a die altered the expectation.

This sets a limit that can at least be located. The number of events remaining before the gap closes is $N_{rem} = GC/q$, which falls to zero as the gap does, so below

$$G \lesssim q/C$$

there are too few events left for a rate to be defined at all. **This is the truncation of Section 4.1 once more, with the grain of the transfer in place of the resolution of an instrument,** and in the circuit it is likely to be the binding one: a delivery of tens of nanocoulombs into a few microfarads puts it in the millivolts, orders of magnitude above the comparator offset and far above the thermal fluctuation. **Where the limit lies is therefore calculable; what the system does below it is not described here.**

The general point is that the continuous description assumes a continuity the systems do not have. The equation of Section 1 is exact for a quantity that flows; every realization in this paper instead transfers in units, whether of charge, of collisions, of slips, or of anything else countable. **Where the units are many the description is excellent and where they are few it has no content,** and the transition between the two happens within every run rather than between runs. **No formulation covering both is offered here,** and the absence is worth stating plainly rather than leaving to be discovered.

The simulations are better placed in this respect, and not by design. They advance the state in discrete steps and transfer in discrete amounts, so they represent the granularity rather than averaging over it; what they cannot represent is a system whose grain is finer than their step size, which simply reappears as the same problem at another scale. **A reader comparing the figures with the closed-form results should expect them to agree where events are dense and to part company where they are not,** and the second case is the one the terminal stretch of every run falls into.

8.5 A Candidate Formalism: State-Space Inversion and Discrete Interventions

Reformulation of Superlinear Feedback via State-Space Inversion: The Reciprocal Gap Formalism. The classical differential representation of superlinear runaway, $dy/dt = k(1 + y^m)$, relies on a continuous, dimensionless feedback exponent m that conceals the underlying geometry of the physical state space. A structurally grounded alternative replaces the runaway observable with the closure of a finite physical gap $a(t) > 0$ (representing a threshold window, remaining liquidity, or spatial clearance).

Let the observable frequency $f(t)$ be defined directly by the reciprocal of the gap under a constant traversal velocity v (normalized without loss of generality to $v = 1$):

$$f(t) = \frac{1}{a(t)}$$

Differentiating $f(t)$ with respect to time yields:

$$\frac{df}{dt} = -\frac{1}{a^2} \frac{da}{dt}$$

To establish the formal correspondence with an autonomous superlinear feedback law $df/dt = c \cdot f^m = c \cdot a^{-m}$ (where $c > 0$), equate the two expressions:

$$-\frac{1}{a^2} \frac{da}{dt} = c \cdot a^{-m} \Rightarrow \frac{da}{dt} = -c \cdot a^{2-m}$$

This identity isolates the exact physical meaning of the dimensionless feedback exponent: **m is not an arbitrary mathematical parameter, but an index governing how the gap closing velocity couples to the instantaneous gap width.**

1. Linear Closure ($m = 2$)

When $2 - m = 0$, the gap closes at a strictly constant rate:

$$\frac{da}{dt} = -c \Rightarrow a(t) = c(t^* - t)$$

which yields the canonical hyperbolic singularity:

$$f(t) = \frac{1}{c(t^* - t)} \propto (t^* - t)^{-1}$$

In hardware, this corresponds to charging an accumulation capacitor with a constant current source.

2. Per-Event Depletion ($m = 3$)

When each cycle removes a discrete quantum δa from the gap, the consumption rate is proportional to frequency:

$$\frac{da}{dt} = -\delta a \cdot f = -\frac{\delta a}{a}$$

Here $2 - m = -1$, giving $m = 3$. Integrating $a \, da = -\delta a \, dt$ yields:

$$a(t) = \sqrt{2\delta a(t^* - t)} \Rightarrow f(t) \propto (t^* - t)^{-1/2}$$

3. Proportional Relaxation ($m = 1$)

When $da/dt = -c \cdot a$, the closing velocity vanishes with the gap. The solution is $a(t) = a_0 e^{-ct}$, so $f(t) = f_0 e^{ct}$. The singularity is pushed to infinity ($t^* \rightarrow \infty$), defining the boundary between power-law drift and finite-time blow-up.

4. General Integration ($m \neq 1$)

Integrating $a^{m-2} da = -c \, dt$ from $a(t)$ to 0:

$$\int_{a(t)}^0 u^{m-2} du = -c(t^* - t) \Rightarrow a(t) = [c(m-1)(t^* - t)]^{\frac{1}{m-1}}$$

Inverting back to frequency directly recovers the classical blow-up scaling exponent:

$$f(t) = [c(m-1)(t^* - t)]^{\frac{-1}{m-1}}$$

* Quantized Interventions and Inherent Convexity.

The operational advantage of the gap formulation is that a discrete, non-infinitesimal shock has an exact effect rather than an

approximate one. Consider a state-independent reduction of the gap by a quantum $\Delta a > 0$, applied at some instant whose timing is not at issue here:

$$a \mapsto a - \Delta a$$

The corresponding change in observable frequency evaluates algebraically:

$$\Delta f = \frac{1}{a - \Delta a} - \frac{1}{a} = \frac{\Delta a}{a(a - \Delta a)}$$

For small discrete increments ($\Delta a \ll a$), this transformation simplifies to:

$$\Delta f \approx \frac{\Delta a}{a^2} = f^2 \cdot \Delta a$$

This reveals that **superlinear sensitivity is entirely structural**:

-In the early regime where the gap is wide ($a \gg 1, f \ll 1$), an intervention Δa generates a negligible perturbation in frequency ($\Delta f \approx 0$).

-In the late regime where the gap is depleted ($a \ll 1, f \gg 1$), the exact same shock Δa produces a violent, non-linear explosion in frequency ($\Delta f \rightarrow \infty$).

The system exhibits extreme, asymmetric sensitivity to discrete perturbations purely as a geometric consequence of state inversion. **This accounts for the granularity of the transfer without appealing to the exponent**: a quantum of fixed size has a negligible effect early and a violent one late, and the exponent is not required to change for this to happen.

* Physical Regularization via Bounded Thresholds.

In the continuous formulation $dy/dt = k(1 + y^m)$, the approach to a singularity produces an unphysical divergence $y \rightarrow \infty$, requiring arbitrary computational cutoffs. Under the reciprocal gap formulation, the terminal event is localized geometrically at $a \rightarrow 0$.

In any physical realization, the gap cannot shrink indefinitely. There exists an intrinsic resolution floor $a_{min} > 0$ defined by the hardware substrate:

$$a_{min} = \max(V_{os}, \sigma_v)$$

where V_{os} is the comparator input offset voltage and $\sigma_v = \sqrt{k_B T/C}$ is the irreducible thermal noise floor. When $a(t) \leq a_{min}$, the threshold ordering becomes indeterminate. The comparator loses arbitration capacity, the oscillation stops, and the observable frequency detaches naturally without invoking mathematical infinities.

What this formulation settles and what it leaves. It settles the relation between granularity and the exponent: a transfer of fixed size has an effect that scales as f^2 , exactly, and the exponent is revealed as an index of how the closing velocity couples to the gap rather than as a parameter imposed on the dynamics. **It also places the truncation geometrically**, at a resolution floor in the gap, where the continuous formulation required an arbitrary cutoff in a diverging quantity.

The formulation extends to a chain, and doing so turns the question of how often transfers occur into a question of how many are required. Index the levels from slowest to fastest, writing a_i for the gap of level i , C_i for its accumulating capacity, and q_i for the quantity each completion of the level above delivers into it. **A completion does not reset the level to zero**: it returns to the level immediately beneath, so the gap of level i is measured from the threshold down to the state of level $i-1$,

$$a_i = V_{thr} - V_{i-1}$$

and the number of completions above required to close it follows immediately. Each delivers q_i/C_i of gap, so

$$N_i = \frac{a_i C_i}{q_i}$$

This is not a constant. Since V_{i-1} only rises, a_i only falls, so N_i falls with it: each successive completion of level i requires fewer events above than the one before. **The acceleration of a cascade is therefore a contraction of a requirement rather than an increase in a rate**, which is the counterpart in event counts of the shrinking $\tau^*(Y)$ of Section 4.2.2.

Composing the chain multiplies rather than adds. One completion of level i costs N_i completions of level $i+1$, each of which costs N_{i+1} of the level above that, so the number of events at the fastest level required to close the slowest gap is

$$N_{total} = \prod_{i=1}^N \frac{a_i C_i}{q_i}$$

which grows geometrically with the depth of the hierarchy. A chain of four levels each requiring a thousand events above it needs 10^{12} events at the fastest level, and this is why the slow levels of the figures appear motionless while the fast ones are a blur: the ratio is structural and not a choice of display.

The hierarchy exists only while each level costs more than one event. Where $q_i \geq a_i C_i$ the count falls below unity and a single completion closes the gap beneath outright; **the level then has no independent existence** and the chain shortens by one, its delivery passing directly to the level below. This is the cascade of collapses described in Section 5.1, obtained here by counting rather than by simulation. **The weights may be taken equal for analysis**, since nothing above depends on their differing, but they decide whether a given realization has a hierarchy at all.

One consequence of counting is that the gap does not close at zero, and the amount by which it misses can be calculated. The last delivery before closure arrives when the remaining gap is somewhere between 0 and q/C , with no reason to prefer any part of that interval, so the overshoot is distributed uniformly:

$$\varepsilon \sim U(0, q/C) \quad E[\varepsilon] = q/2C \quad \sigma_\varepsilon = q/(2\sqrt{3} C)$$

The closure therefore occurs at a gap of order q/C and not at zero, and the terminal state overshoots the threshold by that amount. **This is the same limit Section 8.4 obtained from the other direction.** There the argument was that a rate cannot be defined once few events remain, and the bound was $G \lesssim q/C$; here it is that the last event cannot land precisely, and the bound is the same quantity. **Two independent routes to one limit**, one from the continuous description losing content and one from the discrete description overshooting, which is some reason to think the limit is real rather than an artifact of either.

It also puts a floor under the reproducibility of the closure time. With N_{total} events in a run and an overshoot of this kind at each level, the relative spread of the closure time falls as the count grows: a deep hierarchy with many events closes at a well-determined moment, and a shallow one with few does not. **The same quantity therefore sets the resolution of the description and the reproducibility of the event**, and both improve together as the transfers become finer.

What remains open is narrower than before. The counts above are exact and require no distribution: they follow from the capacities and the quantum alone. **What they do not give is the interval between events**, which depends on how long each instance takes to run and therefore on the dynamics of the level above. **The law $da/dt = -c a^{2-m}$ remains assumed from the continuous side rather than derived from the discrete one**, and deriving it would mean treating the closure as a renewal process whose inter-arrival law depends on the current gap. **That derivation is not attempted here**, and the formulation above says how many transfers a closure takes and how much each one does, but not when they arrive.

The chain written out. Reading from the slowest level to the fastest, with C_i the capacity of each accumulator and P_i the quantity one completion of the level above delivers into it:

$$C_1 \xrightarrow{P_1} C_2 \xrightarrow{P_2} C_3 \xrightarrow{P_3} \dots \xrightarrow{P_{N-1}} C_N \leftarrow \text{base fluctuation}$$

Each link carries a count and each node carries a gap. Level i closes after $N_i = a_i C_i / P_i$ completions of the level to its right, and the fastest level at the right-hand end is driven by whatever supplies the base fluctuation, which is the open question of Section 7.1. **Reading the chain from right to left is reading it in time**: events at C_N are frequent and small, events at C_1 are rare and decisive.

A worked instance makes the scales concrete. Take four identical accumulators of 1 μF , each gap initially 1 V, and each completion delivering 10 nC:

$$N_i = \frac{(1 \text{ V})(1 \mu\text{F})}{10 \text{ nC}} = 100 \quad N_{total} = 100^4 = 10^8$$

so a hundred million events at the fastest level correspond to one closure at the slowest, and to a hundred at the level just above it. **The overshoot is $P/C = 10 \text{ mV}$ at every level**, which is where each gap actually closes and below which no rate is defined. **Adding a fifth level multiplies the event count by another hundred and changes nothing else**, which is what makes the slow levels of the figures look like straight lines.

The same arithmetic gives the condition for the chain to hold together. A level exists as a level only while $N_i > 1$, that is while $P_i < a_i C_i$; in the example a delivery of 1 μC rather than 10 nC would give $N_i = 1$ and the level would be closed by a single event from above, collapsing into the

one beneath it. **Since a_i falls as the structure accumulates, this condition is eventually violated at every level**, and the chain shortens from the fast end as it proceeds.

8.5.1 The Chain in General Form

Nothing in the preceding account is electronic. The circuit supplied convenient names, and they are worth replacing with the quantities they stand for, since the structure is what carries over to any candidate.

Three quantities per level, and one relation between them. A level i has a **gap** a_i , the distance remaining to its threshold in whatever quantity the level is measured in; a **capacity factor** C_i , which converts that distance into the accumulating quantity, so that $a_i C_i$ is the total still required to close the gap; and a **transfer** P_i , the amount of that same quantity deposited by one completion of the level above. **The two conversions must agree:** $a_i C_i$ and P_i are in the same units, which is what makes their ratio a count. The chain is then

$$C_1 \text{ --- } P_1 \text{ --- } C_2 \text{ --- } P_2 \text{ --- } \dots \text{ --- } P_{N-1} \text{ --- } C_N \leftarrow \textit{base fluctuation}$$

with the counts and the composition exactly as before,

$$N_i = \frac{a_i C_i}{P_i} \quad N_{total} = \prod_i N_i \quad \varepsilon_i \sim U(0, P_i/C_i)$$

and three conditions that decide whether a candidate has this structure at all. The transfers must be positive, so that the quantity accumulates rather than circulating; each must be smaller than the capacity it enters, $P_i < a_i C_i$, or the level it enters has no independent existence; and a completion must return a level to the state of the one beneath rather than to its origin, which is what makes a_i contract. **Where all three hold, everything above follows by counting.**

What the three quantities are is a question for each field. In a loaded solid the capacity might be what a portion can bear, the gap what it can still take, and the transfer what a neighbouring failure hands to it; in a market, the capacity of a holder, the amount they will still absorb, and what one liquidation passes to the next. **Identifying them is the work the construction cannot do**, and it is the same identification the membership conditions of Section 4.2 ask for, stated in units a given field can measure.

What remains open is narrower than before, and belongs here rather than earlier. The counts are exact and require no distribution: they follow from the capacities and the transfers alone. **What they do not give is the interval between events**, which depends on how long each instance takes to run and therefore on the dynamics of the level above. **The law $da/dt = -c a^{2-m}$ is still assumed from the continuous side rather than derived from the discrete one**, and deriving it would mean treating the closure as a renewal process whose inter-arrival law depends on the current gap. **That derivation is not attempted here:** the formulation says how many transfers a closure takes and how much each one does, and not when they arrive.

The question does not stop at one level, and that is the point. The rate at which level i receives transfers is set by how fast the level above runs, which is set in turn by the transfers it receives, and so on upward. **Each level answers the question for the one beneath it and reopens it for itself**, so the chain does not close: at the fastest level there is nothing above, and what drives it must come from outside the structure. **This is the question of Section 7.1 arrived at by a different route**, and the route shows it to be structural rather than a gap in the exposition.

It bears on the origin of the fluctuation and not on the theorem. Assumption (S3) concerns how an exponent process behaves, not where it comes from, and is unaffected by any of this; the conditions of Section 4.2 likewise ask what a fluctuation does rather than what produced it. **What is unanswered is prior to both:** in every candidate examined here something at the finest scale is in motion, and the construction takes that as given because it has no means of accounting for it. **That it is present in the systems concerned is not in doubt**, and why it is present is not a question this work can reach.

8.5.2 A Question the Formulation Raises

The accounting above treats the transfer as fixed, and nothing requires it to be. If P_i itself varies as the gap closes, the counts and the overshoot change with it, and the exponent recovered from the record changes with them. **Whether a varying transfer can reproduce what a varying exponent produces is an open question**, and it is a sharper one than it looks: the two act in different places, and a record showing one cannot be assumed to indicate the other.

The question has a specific form worth stating. Section 3.1 shows that deep subcritical excursions flatten the early record toward a power law while the system still terminates; the mechanism there is a fluctuating exponent. **A transfer that is small early and grows as the gap closes would produce a record of the same shape**, with the exponent fixed throughout. If so, the masking of Section 3.1 is not peculiar to exponent fluctuation but is available to any system whose grain is not constant, which would widen that result considerably and weaken what can be read from a flattened record.

This is tractable and is not pursued here. It requires no new construction, only the accounting of Section 8.5 carried through with P_i a function of a_i , and it is testable on the circuit of Section 6, where the transfer is set by the controller and can be varied deliberately while the exponent is held fixed. **It is recorded as a direction because it belongs to the mathematics of granular transfer rather than to the isomorphism this paper is about**, and because the answer, either way, would bear on how a flattened record should be read.

8.5.3 The Exponent as a Curve, and the Regress Beneath It

Section 8.4 asked when the exponent changes and found no answer. The accounting of Section 8.5 gives one, and it is not the expected one.

A transfer is not instantaneous: in the circuit the winding is energized for an interval t_{on} , the field builds, and the charge arrives over a further interval. **During that interval the gap is closing at a rate that is neither zero nor its between-event value**, so the exponent is not a number that switches but a quantity defined over the interval, rising and falling with the transfer.

The exponent is therefore a curve, even where the record is a staircase. It is tempting to read a stepped record as an exponent that is off and then on, taking one value between transfers and another during them; this is not what the quantity does. **What switches is the delivery; what the exponent does is follow it**, with a shape set by how the transfer rises and decays. A two-valued description is an approximation to that curve, adequate where the intervals are short compared with everything else and misleading where they are not.

The limit $t_{on} \rightarrow 0$ does not exist, and its absence is informative. A transfer completed in no time would require the exponent to be unbounded for no duration, which is the same singularity the construction describes, now appearing in the description of a single event. **A finite transfer interval is what makes the exponent a well-defined quantity at all**, in the same way that a finite resolution floor is what makes the gap one. The two are the same kind of limit at opposite ends of the same process.

This also makes the drift of Section 8 calculable in advance rather than merely expected. Where the period approaches t_{on} the transfers cease to be separate events and become a continuous supply, which is the passage from per-event to per-unit-time delivery and therefore from $m = 3$ toward $m = 2$. **The crossing lies near $f \approx 1/t_{on}$** , and since t_{on} is set by the experimenter, the location of the drift is a design parameter. **A prediction of this kind is worth more than a fitted exponent**, since it can be moved deliberately and checked.

Whether the curve is itself smooth depends on the system, and where it is not the argument repeats. If the transfer is composed of smaller discrete deliveries, then resolved finely enough the rise of the exponent during an interval is again a staircase, and everything above applies one level down with a smaller quantum and a shorter interval. **This is a property of particular systems and not a general necessity**: a transfer carried by a quantity with no evident grain at the next scale admits no such refinement, and whether a given system has one is a question for the field concerned rather than for this construction.

Where the refinement does repeat, it terminates where the grain does. In the circuit the next step down is thermal, at the scale of Section 6.3, and what lies below that is not something this paper can speak about. **The consequence for the formulation is specific and should be stated as such**: an exponent in this construction is defined relative to a window, and the smallest admissible window is set by the grain of whatever is transferring. **Below that there is no exponent to speak of, not an unmeasured one**, which is the same conclusion Section 8.4 reached for the gap and Section 7.1 leaves open for the fluctuation that drives it.

8.5.4 A Conserving Cascade

The accounting of Section 8.5.1 leaves the transfer as a free parameter, and one choice removes that freedom. Suppose a completing instance delivers not some designated quantum but everything it holds, so that what the level above has accumulated passes in its entirety to the level beneath. **The transfer is then determined by the state of the system rather than imposed on it**, and the chain conserves along its length.

The consequence is that the quantum is no longer fixed. A fast instance begins at the level beneath it and runs to the threshold, so the span it covers is the slow gap itself, $\Delta V = a_i$, and what it delivers on completing is

$$P_i = C_{i+1} a_i \quad \rightarrow \quad \Delta a_i = P_i / C_i = r a_i, \quad r = C_{i+1} / C_i$$

so each event removes a fixed fraction of what remains rather than a fixed amount. The hierarchy is then carried by the capacities: with r of order 10^{-1} to 10^{-2} , a level is small compared with the one beneath it, and the ordering of Section 4.3 follows from that alone.

The count that caused the difficulty of Section 8.4 is now invariant. Substituting into $N = a_i C_i / P_i$,

$$N = \frac{a_i C_i}{C_{i+1} a_i} = \frac{1}{r}$$

which does not depend on the gap. At a tenth of the capacity there are ten events remaining whatever the distance left, and the same ten at a hundredth of that distance. **The situation in which too few events remain for a rate to be defined does not arise**, because the grain contracts

with the gap it is dividing. This is the difficulty of Section 8.4 removed rather than bounded, and removed by a choice of coupling rather than by a further assumption.

The exponent that results is $m = 2$, and it does not depend on the ratio. The fast instance starts from the slow level, so its completions arrive at a rate $v \propto c/a$, and each removes $r a$:

$$\frac{da}{dt} = -(r a) \frac{c}{a} = -r c \quad \rightarrow \quad a(t) = a_0 - r c t \quad \rightarrow \quad m = 2$$

The gap closes linearly, in a finite time $t^* = a_0/(rc)$, through infinitely many events. Since the fraction removed is constant, the events form a geometric sequence and accumulate at the closure without ever exhausting it, which is the structure of an inelastic collapse rather than of a last decisive step. **The capacity ratio sets how fast this happens and not what happens:** r appears in the rate and nowhere in the exponent.

That the exponent should come out at 2 is worth noting rather than passing over. By Section 5.2 it is the unique value a driver of bounded capacity can sustain to the end: below it the demand falls away, above it the demand diverges. **A cascade that conserves what it transfers arrives at that value without being set to it**, which is a different route to the same place and was not anticipated when either result was obtained.

What this settles and what it does not. It settles the granularity: the transfer scales with the gap, so the description retains its content at every scale and the truncation of Section 8.4 does not occur. **It does not settle the origin of the fluctuation**, which is the question of Section 7.1 and is untouched by any of this; what it does is reduce the scale at which that question has to be answered, since an arbitrarily small disturbance at the top of a conserving chain is sufficient. **And it holds only while the contraction continues:** any physical realization has an absolute floor, thermal or otherwise, below which the quantum no longer shrinks with the gap, and beneath that floor the difficulty of Section 8.4 returns in its original form.

As a basis for simulation this is the better construction. It has one parameter per level rather than two, it obeys a conservation law that can be checked at every step, and it has no cutoffs to set, since the quantities rescale rather than terminating. **The simulations of this paper do not use it**, having been written before it was obtained, and a reader rebuilding them on this basis would be improving them in a way the author has not.

8.5.5 Where a Conserving Cascade Actually Stops

The cascade of the preceding subsection does not end of its own accord, and this makes the question of where it ends unusually clean. Each event removes the fraction r of what remains, so the gap follows a geometric sequence

$$a_n = a_0 (1-r)^n$$

which approaches zero without reaching it at any finite n . **No event overshoots**, since each is proportional to what is left, and the overshoot of Section 8.5 therefore has no counterpart here. The closure is an accumulation point of infinitely many events rather than the arrival of a last one, which is why the time to it is finite while the count is not.

Any truncation must therefore come from outside the construction. With an additive fluctuation of width σ in the gap, of the kind Section 6.3 describes, the sequence proceeds undisturbed while $a_n \gg \sigma$ and ceases to be meaningful once

$$a_n \approx \sigma \quad \rightarrow \quad n^* \approx \frac{\ln(a_0/\sigma)}{\ln(1/(1-r))}$$

and the logarithm makes this a small number. With $r = 0.1$ and a fluctuation six orders of magnitude below the initial gap, $n^* \approx 131$; with $r = 0.5$, about twenty. **An infinite sequence is cut to a few dozen terms by a fluctuation that is negligible at every point until the last few**, and the count depends only logarithmically on how small the fluctuation is, so reducing it by further orders of magnitude buys very little.

The division of labour is complete and worth stating as such. The structure supplies the dynamics and sets no limit of its own; the fluctuation supplies the limit and plays no part in the dynamics. **In the fixed-quantum case of Section 8.5 the two were mixed**, since the overshoot of the last transfer and the external noise both acted near the closure and neither was separable from the other. **Here the overshoot is absent and the truncation is entirely the fluctuation's**, which means its location is a measurement of σ rather than of anything about the cascade.

This sharpens the question of Section 7.1 rather than relieving it. A conserving cascade has no scale of its own: the ratio r fixes the rate and the geometry repeats at every magnification, so nothing in the construction says where the process should stop. **Everything that does say so comes from the fluctuation**, and the fluctuation is the one element this paper cannot account for. **The better the structure is understood, the more of the remaining explanation falls to the part that is not.**

8.5.6 The Fixed-Quantum Cascade Is Chaotic, and Why

The conserving cascade of Section 8.5.4 and the fixed-quantum cascade of Section 8.5 differ in more than their exponents. They differ in how a disturbance propagates, and in opposite directions: one absorbs it and the other amplifies it. **The difference follows from a single line of arithmetic** and is worth setting out, since it decides whether a given system has a reproducible terminal state.

Take the fixed-quantum case, in which each completion above delivers a quantity q and the gap descends by steps of q/C . Starting from a_0 , the gap after n events is $a_0 - n q/C$, and closure occurs at the first n for which this is not positive. **What is left over at that moment is not zero but a remainder:**

$$\varepsilon(a_0) = a_0 \bmod (q/C)$$

and this remainder is what the level beneath receives as the opening condition of its own descent. The quantity passed down is therefore not the initial gap but its residue modulo the quantum, and the same operation is applied again at the next level with that level's own quantum.

The map is not continuous, and that is the whole of the result. A perturbation δ applied to the opening gap gives a new remainder $(a_0 + \delta) \bmod (q/C)$, which equals $\varepsilon + \delta$ while the sum stays below the quantum and drops by a full quantum as soon as it does not. **The derivative is unity almost everywhere and undefined on a set of points spaced q/C apart**, so an arbitrarily small displacement can move the terminal state by the entire quantum. This is the standard sawtooth, and iterating it is the standard route to sensitive dependence.

Along a chain the iteration is what matters. Writing ε_i for the residue passed from level i to level $i+1$,

$$\varepsilon_{i+1} = \varepsilon_i \bmod (q_{i+1}/C_{i+1})$$

and each application discards the integer part and keeps the fraction. Where the quanta are incommensurate with the gaps, the sequence of residues is as irregular as the initial value is precise: a disturbance invisible at the first level determines which side of a boundary the second lands on, and the third inherits that choice magnified. **The terminal state of a chain of any depth is therefore not predictable from a measurement of finite precision**, and no noise is required for this; the arithmetic supplies it.

The conserving cascade does the opposite, and the comparison makes the mechanism plain. There each event removes a fraction r of what remains, so a perturbation δ becomes $\delta(1-r)^n$ after n events and decays geometrically; there is no remainder, no boundary to cross, and nothing to amplify. **The same structure is therefore stable under perturbation when the transfer is proportional and chaotic when it is fixed**, and which of the two a system does is a question about its coupling rather than about its dynamics.

A symmetry is broken here, and it is worth naming because it is not obvious from the equation. The continuous law $da/dt = -c a^{2-m}$ is invertible: given the state at one moment, the state at any earlier moment follows, and the trajectory can be run backward as readily as forward. **The fixed-quantum map is not.** The operation $a \rightarrow a \bmod (q/C)$ discards the integer part, and no inverse recovers it: a residue of 0.3 quanta is consistent with an opening gap of 0.3, 1.3, 2.3 and so on without limit. **Infinitely many pasts lead to the same present**, and the information distinguishing them is destroyed at each level.

The continuous description therefore has a time-reversal symmetry that the discrete process does not, and the symmetry is broken by the granularity rather than by any dissipation. This is not a defect of the discrete account but the reason the two accounts disagree about predictability: the smooth law permits reconstruction of the past and prediction of the future from any single state, and the stepped process permits neither beyond the resolution of one quantum. **Where a candidate system transfers in fixed amounts, a terminal state reproducible to better than one quantum should not be expected**, and a measurement finding one would indicate that the transfer is proportional rather than fixed.

One clarification, since the vocabulary invites a misreading. What fails here is the invertibility of a map and nothing more. **Time-translation invariance is untouched:** neither the continuous law nor the discrete process depends on when it is started, and the same initial state gives the same evolution whenever it is set going. **Nor is Noether's theorem in question**, which applies to systems derived from an action principle; the equation of Section 1 is first-order and dissipative, has no Lagrangian, and lies outside its scope entirely.

What is lost is information, and the loss is exact. The map $a \rightarrow a \bmod (q/C)$ is not injective: it discards the integer part, and the integer part cannot be recovered from the residue. **Each level of a chain therefore destroys a definite quantity of information**, namely which multiple of the quantum the gap had contained, and after a few levels nothing of the original state survives except a fraction of one quantum. The continuous law destroys nothing and can be integrated backward; the stepped process cannot. **Sensitive dependence here is a consequence of that non-invertibility rather than of any instability in the dynamics**, and the two should not be conflated.

9. A NOTE ON SCALE

One property of the construction deserves comment, because it invites a stronger reading than it supports. **The equation has no intrinsic scale.** By Proposition 0, the rate sets only the clock and the mapping constant only the unit, so nothing in the mathematics fixes a magnitude, a duration, or a

domain at which it is meant to apply. The same is true of the nested form: the construction of Section 5 can be extended downward to faster instances or upward to slower ones without any term acquiring a preferred size.

This is a property of the equation, and it is not evidence about the world. Scale invariance in a model means that the model does not select a scale; it does not mean that the systems the model might describe are related to one another across scales. The membership conditions of Section 4 apply at every level independently: a candidate at any scale must have its governing equation reduced to this form, its non-decreasing state variable identified, and the quantity that moves its exponent established. **Nothing is inherited from one scale to another, and a correspondence found at one says nothing about any other.**

The temptation the absence of scale creates should therefore be named and set aside. **A framework that can be written at any scale is not thereby a framework that connects scales**, and the distance between the two is the entire content of the membership conditions. This paper makes the first claim and not the second.

A related observation points the other way, and is worth recording because it costs nothing. Systems showing similar behaviour at several scales are often described as fractal, and the description is usually a report of what was seen rather than an account of why. **This construction supplies one possible account.** If several instances of the same equation run at different rates, their trajectories have the same shape by Proposition 0, since the rate sets only the clock; resemblance across scales then follows from a shared equation and does not require self-similar structure, a non-integer dimension, or anything else the term ordinarily implies.

What recommends this reading is that it also accounts for the differences. Exact self-similarity predicts that the scales match; it has nothing to say when they do not. **Here the departures are expected and their sources identified:** the exponent drifts for the reasons of Section 8, the depth of the hierarchy grows as Section 4.2.2 describes, and the cutoffs differ from level to level. **A system that resembles itself across scales but not exactly is awkward for one description and ordinary for the other**, which is a place where the two could in principle be told apart.

None of this privileges the equation used here. What the argument needs is that one law operates at several rates, not that the law is this one; another equation with the same property would produce the same resemblance. **The claim is therefore about an explanation being available rather than about this explanation being right**, and the same caution applies to it as to everything else in this section.

One question is worth leaving open, stated as a question. In the construction as it stands, an instance that completes its asymptote leaves the observation: the moving average detaches from it and falls back onto the structure beneath. **The construction says what happens to the reading. It does not say what happens to the instance.** That silence is worth noticing, because it is not the silence of something that has been ruled out.

Two facts make the question well posed rather than idle. **The first is that nothing is lost at the detachment.** The moving average remains finite through it, not because the instance beneath it stops, but because the logarithm of a function diverging at a finite point is integrable; the instance completes its asymptote in full while the reading of it stays bounded. **The detachment is a property of the observation, not an event in the thing observed.** The second is that the completion occurs at a point. A finite-time singularity is reached in finite x , so whatever the completed instance amounts to, **it is brief when measured on the clock of any slower level**, by the factor separating their rates.

The question, then, is this. Suppose one sets aside the structure the instance grew out of, and asks only what a completed instance is, viewed from a slower level. It is a quantity that arrives, is enormous on its own scale, and is over in an instant on the slower one. **On that level it has the form of a fluctuation.** And a fluctuation, in this framework, is not a disturbance but the thing that moves the exponent, and hence the thing from which a sequence of instances can begin. **Whether a completed instance can serve in that role, becoming the starting point of a new structure rather than only the end of an old one, is not answered here.**

What this paper does not ask, deliberately. How such an event would act on the slower structures already present, whether it would be absorbed, amplified or ignored, is a question of coupling and of parameters, and is left alone. So is any question of what the completed instance consists of, which the mathematics does not address. **The question retained here is narrower and structural:** whether the construction admits a second kind of transition alongside the one it describes, in which a completion is not only an ending in one place but an origin in another.

This is recorded as a question and nothing more. It is not a conjecture in the sense of Section 4, since no observation is proposed that would bear on it and no candidate system is named. **It is placed in this section, among the things the absence of scale does not license, precisely so that it is read as a question about the mathematics rather than as a claim about the world.**

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Physical Candidates and Isomorphisms

1. POSSIBLE CONNECTIONS: TOKAMAK EDGE INSTABILITIES AND METASTABLE ESCAPE

The systems examined so far were candidates for the mechanism of Part II, Section 3: an exponent that fluctuates across the critical boundary. **Tokamak edge localized modes (ELMs), as they are currently described in the literature, are not such a system, and the purpose of this section is to say precisely why, and what the established description amounts to instead.** That literature derives a genuine finite-time singularity from first principles; but in the derivations available at the time of writing, **the nonlinearity is quadratic and fixed**, arising from the second term of a Taylor expansion about marginal stability rather than from any stochastic quantity. What varies is the linear coefficient and the size of the perturbation that arrives. On that description, ELMs fall into a third class, **metastable escape**, which this section introduces and which the remainder of the paper treats as distinct from both exponent noise and prefactor noise.

The generalized equation. To accommodate this class, the equation of Part II, Section 1 requires one additional term, a linear damping:

$$\frac{dy}{dx} = -a(x)y + ky^m$$

The constant term of the original equation is dropped, because here the origin is an equilibrium rather than a driven state. **The sign of a decides the character of the system.** For $a < 0$ the origin is unstable and every trajectory departs it. For $a > 0$ the origin is *linearly stable*, yet for $m > 1$ the superlinear term still dominates at large amplitude, so **the system is stable against infinitesimal perturbations and unstable against sufficiently large ones**. This is metastability, and the original equation of Part II, Section 1 cannot represent it: with a positive constant term, every trajectory grows from the start and no threshold exists.

The threshold and the escape time. For $m = 2$ the equation is separable and yields a closed solution. Writing the threshold amplitude as $y_c = a/k$:

$$y(x) = \frac{a}{k - (k - a/y_0)e^{ax}} \quad x^* = -\frac{1}{a} \ln(1 - y_c/y_0)$$

For $a < 0$ the blow-up time is finite for every initial amplitude. **For $a > 0$ it is finite if and only if $y_0 > y_c$** ; below the threshold the trajectory decays to zero. Two features are worth noting, because both reappear in the plasma. The escape time diverges logarithmically as the initial amplitude approaches the threshold from above, so **events triggered just above threshold are slow and those triggered well above it are abrupt**. And as a is driven toward zero, the threshold y_c falls in proportion, so **an ever smaller perturbation suffices**.

The isomorphism. Nonlinear ideal ballooning theory describes the eruption of a plasma flux tube. In the drag-dominated limit, Cowley, Cowley, Henneberg & Wilson (*Proc. R. Soc. A* 471, 20140913, 2015) obtain for the displacement ξ of a field line

$$(v_D/\rho_0) \frac{d\xi}{dt} = \Gamma_0^2 \xi + C_4 \xi^2$$

where v_D is a drag coefficient, ρ_0 the mass density, Γ_0^2 the local linear drive and C_4 the coefficient of the explosive nonlinearity. **Dividing through by v_D/ρ_0 gives exactly the equation above**, under the correspondence

$$y \leftrightarrow \xi \quad a \leftrightarrow -\Gamma_0^2 \rho_0 / v_D \quad k \leftrightarrow C_4 \rho_0 / v_D \quad m = 2$$

The correspondence is exact, not approximate, and the plasma literature derives the same two consequences independently. The critical amplitude for a linearly stable field line is $\xi_c = |\Gamma_0^2|/C_4$, which is the threshold $y_c = a/k$ above; and in the linearly unstable case the eruption occurs at $t_\infty = (1/\gamma) \ln(1 + \xi_c/\xi_0)$, which is the escape time above with $a = -\gamma$. **The tokamak edge, in this limit, is computing the same equation.**

Where the singularity comes from, and why the exponent is fixed. The general theory (Cowley & Artun, "Explosive instabilities and detonation in magnetohydrodynamics", *Phys. Rep.* 283, 185, 1997; Wilson & Cowley, "Theory for Explosive Ideal Magnetohydrodynamic Instabilities in Plasmas", *Phys. Rev. Lett.* 92, 175006, 2004) expands the energy functional about the marginal state. The quadratic term of that expansion is the explosive nonlinearity, so **$m = 2$ is structural within that treatment: it follows from the expansion itself, and not from any property of the plasma that the treatment allows to fluctuate**. Whether a description in which the effective exponent is not fixed in this way would better match the observations is not a question this paper can settle, and it is left open here. In slab geometry the displacement diverges as $(t_0 - t)^{-2.1}$; numerical solutions of the full equation give exponents near -2.5 . In toroidal geometry the inertia term becomes a fractional time derivative of order λ , set by the Mercier indices, and the divergence exponent becomes λ itself, reducing to $(t_f - t)^{-1}$ for a vanishing Mercier coefficient.

A remark that follows from this, and that applies well beyond the plasma. Balancing a time derivative of order λ against a nonlinearity of order m gives $y \propto (t_0 - t)^{-\lambda/(m-1)}$. Part II, Section 2 is the case $\lambda = 1$. **A measured divergence exponent therefore does not identify m on its own**: the same value is produced by first-order dynamics with one exponent and by second-order dynamics with another. The slab value near 2.1 is consistent with $m = 2$ and $\lambda = 2$, not with a larger nonlinearity. Any attempt to read a feedback exponent from an observed blow-up profile must therefore establish the order of the governing dynamics first. This observation is the present author's; it is not drawn from the plasma literature.

The mechanism, on the established picture: a slow ramp and a sudden kick. Between events the pedestal is heated and fuelled, its pressure gradient rises toward the peeling–ballooning limit, and the linear coefficient drifts toward zero: in the notation above, **$a(x)$ decreases and with it the threshold $y_c = a/k$** . The edge is metastable throughout this phase. Linear instability is not required for eruption, since a finite—though not infinitesimal—perturbation will destabilize a linearly stable profile. The asymmetry involved is severe: in the model atmosphere of Cowley et al. (2015), a critical perturbation of about 1% of a field line's magnetic energy releases roughly 17% of it. In tokamak geometry, Ham et al. (*Phys. Rev. Lett.* 116, 235001, 2016) find metastable saturated states over a window of the normalized pressure gradient lying *below* the linear stability boundary.

This account also offers a reading of why pellet pacing works. Injecting a pellet is an engineered perturbation above the threshold, and it triggers an ELM before the natural ramp would have delivered one. Because the event is triggered while a is still comparatively large, the accumulated pedestal is smaller and the resulting crash is milder; raising the ELM frequency above its intrinsic value was found to reduce the severity of the individual events (Lang et al., *Nucl. Fusion* 44, 665, 2004). **In the framework above, pacing does not alter the dynamics; it chooses the moment of escape.**

Detachment: the singularity is never reached. As Part II, Section 4.1 requires of any physical realization, the divergence is truncated. Several mechanisms are proposed in the literature, and on the standard account they act in sequence. The weakly nonlinear expansion itself breaks down well before the formal singularity. The erupting tube then narrows, twists and either saturates into a finitely displaced equilibrium or is broken up by the sheared edge flow. Finally, resistive reconnection and cross-field heat transport along the erupted tube vent the stored energy. **What is observed is therefore not an infinity but an abrupt, bounded event**: a type-I ELM expels of the order of ten percent of the pedestal energy, strongly dependent on collisionality, after which the gradients recover within a few milliseconds and the cycle repeats at tens of hertz. The correspondence with the regularized-singularity picture of Part II, Section 4.1 is close, and in this respect the plasma is the best-documented example available.

Where the state condition does and does not apply. It would be easy to conclude from the ELM cycle that the non-decreasing state condition fails, since the pedestal rises monotonically only within a single inter-ELM interval and is reset at each crash. **That conclusion would apply the condition at the wrong level.** As Part II, Section 4.2.3 sets out, physical systems of this kind are bounded, the observed quantity cannot pass a certain level, and it must leave the instance it was following and fall back to a lower one, which may be zero. A reset of the observable is therefore *expected* in any bounded realization, and is not by itself evidence against membership. **The condition constrains the latent variable, not the record.**

What is genuinely missing is the identification of that variable. In the established description, the quantity that rises between crashes is the pedestal itself, and that quantity is reset; impurities, so far from accumulating, are expelled by the crashes, which is one reason ELM-free regimes require an alternative exhaust channel. No slower, irreversible quantity has been identified that would play the role of the underlying state in Part II, Section 3. **This is an absence of identification rather than a demonstrated absence,** and the distinction matters: the same would have been said of any candidate system in this paper before its governing equation was reduced. The question is taken up again at the end of this section. A related one is whether the ELM sequence has a terminal member at all, in the sense in which the nested construction ends when its outermost supporting instance completes its own asymptote, and whether such an event would be distinguishable from the other ways a discharge can end. **Neither question is addressed by the literature reviewed here.**

How the sequence is currently described. Independently of the above, the established treatment of the ELM sequence is a renewal process with reset, and the ELM literature has adopted one: under the assumption that the likelihood of an ELM rises monotonically with time since the last, the waiting times follow a Weibull-class distribution, which was confirmed on JET data (Webster & Dendy, *Phys. Rev. Lett.* 110, 155004, 2013). **This description is compatible with both readings,** since a renewal structure with an increasing hazard rate is what a ramped threshold produces and also what a bounded nested structure produces between its terminal events; it therefore does not discriminate between them.

An open question about the reset. If the state were fully reset at each crash, and nothing carried across, the cycle would be governed entirely by the ramp and the noise, and sustained suppression would be a matter of holding the threshold above the ambient perturbation level. In part this is what is observed: suppression over long pulses has been achieved with resonant magnetic perturbations and in quiescent regimes. **It is nevertheless not obvious that a fully reset system would recover its previous behavior as reliably as it does,** and the absence of material accumulation does not by itself establish the absence of accumulation. Candidates for a slower carried quantity would include the evolving current profile, the conditioning of the wall, and resonant structures that persist across events. **This is raised as a question rather than a claim:** within the framework of this paper it is the question of whether an irreversible variable exists at a level slower than the one at which the cycle is observed, which is precisely the variable Part II, Section 4 requires a candidate system to identify.

Three classes, not two. Taking this section together with Part II, Section 3 and Proposition 5, candidate systems divide, *according to the equation currently used to describe them*, as follows. **Prefactor fluctuation:** noise in the rate, which cannot produce a singularity at any amplitude. **Exponent fluctuation:** the mechanism of Part II, Section 3, in which the outcome is decided by the upper tail of the exponent and the mean is irrelevant. **Metastable escape:** a fixed superlinear exponent with a linear damping whose coefficient is slowly driven toward zero, where the outcome is decided by whether a perturbation exceeds a falling threshold. **The three are distinguishable from the governing equations, and that reduction is what assigns a system to one of them.** They are not reliably distinguishable from the shape of a record, and the plasma case shows why: a build-up followed by a collapse is produced by all three. Assigning a candidate to the wrong class on the strength of that shape is the most likely error this framework invites.

What is not claimed. The correspondence established here is between equations, and the plasma satisfies the membership conditions of Part II, Section 4 only within a cycle. **What this section claims is narrower: that the currently established governing equation of the drag-limited plasma eruption reduces to the damped form of this paper's equation with $m = 2$, and that the plasma, as currently modeled, is therefore an instance of metastable escape rather than of intermittent superlinearity.** Nothing here should be read as a claim about what the plasma does, as distinct from what the present models say it does; those models may be superseded, and this paper is itself an attempt to add something to the same body of description. **Where the established treatment is unsettled or disputed within its own field, this paper takes no position,** and the questions raised in the following subsection concern the status of a derived quantity, not the physics from which it was derived.

An observation on prior work. Cross-domain mapping is not new in this field: the same explosive instability has been applied to solar flares, magnetospheric substorms and tokamak disruptions by the same authors, and ELMs have separately been modelled as sandpile and self-organized-criticality systems (Chapman, Dendy & Hnat, *Phys. Rev. Lett.* 86, 2814, 2001). **What appears to be absent is any connection to the general blow-up literature:** the plasma papers derive and use the elementary singularity of $d\xi/dt \propto \xi^2$ without reference to the Osgood criterion, and no link has been drawn to the finite-time-singularity treatments of rupture, or to the accelerating-creep relation of Part I, Section 7, whose inverse-rate forecasting method is the same mathematics applied to a different substance.

1.1 A Caution About the Fixed Exponent

The preceding section takes the exponent of the plasma's nonlinearity as fixed at 2, because that is what the established derivations give. **It is worth being precise about what that establishes and what it does not,** since the classification of the previous section turns on it.

The exponent is derived, not measured. It arises from expanding the energy functional about the marginal state and retaining the leading nonlinear term, which is quadratic. **An expansion of this kind is local by construction:** it holds in a neighbourhood of marginality and for displacements small compared with the width of the unstable region, and the same literature notes that this ordering breaks down before the singularity is reached. What the derivation establishes is that the leading correction near marginality is quadratic. **It does not establish that the effective exponent remains 2 through the phase that produces the observed event.**

The coefficients are not constants, and they are known to vary. Both the linear drive and the coefficient of the nonlinear term depend on the equilibrium; the linear drive varies with the pressure gradient, with sheared flow, and with the turbulence that the previous section describes as shifting the threshold, and the coefficient of the nonlinear term is itself a function of the equilibrium rather than a universal number. **A quantity that depends on the equilibrium is not a constant of the dynamics.**

This matters because a fixed exponent and a varying coefficient are not separate descriptions. For the damped equation of the previous section, the local logarithmic slope of the right-hand side is

$$m_{\text{eff}}(\xi) = \frac{\Gamma_0^2 + 2C_4\xi}{\Gamma_0^2 + C_4\xi}$$

which rises from 1 at small amplitude toward 2 at large amplitude, and which moves whenever the linear drive moves. Whether one says that the exponent is fixed and the coefficient fluctuates, or that the effective exponent fluctuates, is a matter of which variable the equation is written in.

Within the quadratic truncation this slope stays between 1 and 2, so it is not in itself the mechanism of Part II, Section 3; the narrower point stands regardless. **"The exponent does not fluctuate" is a property of how the equation is written, and not an independent fact about the plasma.**

A fluctuating exponent is not excluded by the exponent that governs an eruption. Suppose the effective exponent varied over some range rather than being fixed. By Part II, Section 2, the terminal behaviour of a completing event is set by the value prevailing during the excursion that completes it, and not by the average of the process. **That excursion is necessarily supercritical,** since by the blow-up criterion no subcritical stretch can complete a singularity. The exponent that governs an eruption is therefore a value drawn from above the critical boundary whatever the mean may be, and a value of 2 there is consistent with a fixed exponent of 2 and equally consistent with a fluctuating one whose mean lies at or below 1 and whose range includes 2. **The two descriptions differ in the mean, and the eruption carries no information about the mean.**

A note on the origin of this caution. The model from which this work began also had a fixed exponent, and it was taken as given for the same reason: it fitted the observations closely, and the functional form in which it was written made no other value natural. **It was only later established that the exponent could not be recovered from the observed trajectory at all** (Part I, Section 7.5), and that the fit was consistent with a whole family of values. The exponent was, in the end, a property of the description rather than of the thing described. **That experience is the reason for this section,** and it is offered as a methodological caution rather than as a criticism of the plasma treatment, which is derived carefully and from first principles. The narrow point is this: **an exponent that enters a model by construction is not thereby a measured feature of the system, and is worth measuring independently.**

A test of the metastable reading, and a reason the question is not closed. If the established description is right and these are escapes over a threshold, then **holding the threshold above the ambient disturbance should prevent the escape indefinitely,** and the problem is one of control rather than of accumulation. If instead something accumulates across cycles, as Section 1 raised and left open, then raising the threshold postpones rather than prevents, and suppression should eventually fail however well it is maintained. **The two readings therefore differ in what long-duration suppression would show.**

The evidence available is partial and points both ways. Suppression over long pulses has been achieved by resonant magnetic perturbations and in quiescent regimes, which the metastable reading predicts and which an accumulating quantity would have to accommodate; against that, the suppression requires adaptive feedback rather than a fixed setting, and no device yet sustains the state indefinitely. **Neither observation settles the question,** since a control system that must keep adjusting is consistent with a drifting threshold and also with an operating point that is merely difficult to hold. **What would settle it is whether the adjustment required drifts in one direction over the life of a discharge,** which is a question about control-system logs rather than about plasma theory, and which this paper is not in a position to answer.

Mitigation by pellet injection bears on the same question, and in a way that sharpens it. Injected pellets trigger an eruption earlier than it would otherwise occur, which reduces the size of each event. **Triggering is not the same as removing whatever has accumulated,** and the two readings of Section 1 differ here: if the eruption is an escape over a threshold, an earlier trigger prevents the larger event; if something accumulates in the volume, an earlier trigger discharges only what the pellet reaches, and the remainder continues where it did not.

The test is one of spatial coverage rather than of timing. A pellet affects a region, not the whole plasma, so under the accumulating reading the untouched regions should continue to build and the statistics of events in paced and unpaced regions should differ. **Under the threshold reading they should not,** since what matters there is only whether the threshold has been crossed. The comparison requires spatially resolved measurement rather than the integrated signals usually reported, which is a demand on instrumentation rather than on theory.

A second limit would remain even if coverage were complete. Pellets are discrete, and their size sets a floor on how finely the discharge can be apportioned: below the scale of a single pellet there is no partial action. **That floor does not shrink as the quantity to be discharged shrinks**, so the relative coarseness of the intervention grows as the intervention becomes more effective, which is the situation described in Part II, Section 4.1 for a gap approaching the scale of its contents. **Whether this limit is reached before others is not something this paper can judge**, and it is noted because it is the same structure appearing in a different system rather than because it is expected to be the binding constraint.

A third reading has become available since Section 1 was written, and it resembles the metastable one closely enough to be worth separating from it. Part II, Section 4.2.2 shows that a nested structure recruits levels as it proceeds: an instance too slow to complete within a given window early in a run becomes fast enough once the level beneath it has risen. **The consequence is a long quiet phase followed by an abrupt acceleration**, since the contributors are few at first and multiply toward the end, and a fit to the terminal stretch reports an exponent that drifts upward. **That is also what a metastable escape looks like from the outside**: quiescence while the system sits below its threshold, then rapid growth once it has crossed. A measurement confined to the eruption separates the two no better than it separates either from a fixed exponent.

They differ in the quiet phase rather than in the event, and the difference is one of principle. Below its threshold a metastable system relaxes: the damping term returns it toward a stable point, and whatever accumulated between disturbances is given back. **A recruiting cascade does not relax**, since its counter is a ratchet and the quiet phase is one of slow, monotone accumulation with too few events to register. **The question is therefore whether the quiet interval shows decay toward a stable state or creep away from one**, which is the question Section 1 raised about accumulation across cycles, arriving now from a second direction.

For the pedestal this is not a rhetorical question. Its rebuilding after each crash is consistent with relaxation toward a threshold, which is the established reading and which this paper does not dispute. **What the recruiting account would predict in addition is a quantity that does not return to where it began**, and whose slow accumulation is visible between cycles rather than within them. **The two readings therefore make different predictions about the inter-ELM interval and the same prediction about the ELM**, which is an argument for examining the former, and the same conclusion this subsection reached by its other route.

The two coincide in the terminal stretch, and this can be shown rather than asserted. Writing the metastable equation with its damping term and factoring out the nonlinearity,

$$\frac{dy}{dx} = -ay + ky^m = ky^m [1 - (y_c/y)^{m-1}], \quad y_c = (a/k)^{1/(m-1)}$$

the correction vanishes as a power of y once the threshold is passed, so beyond y_c the equation is the pure superlinear one. In the recruiting case the prefactor instead grows with the support as levels are added; writing that growth as $k_{\text{eff}} \propto y^\beta$ over the relevant range gives

$$\frac{dy}{dx} \propto y^{m+\beta}$$

which is the same family with a shifted exponent. Both therefore terminate as $(x^* - x)^{-1/(m_{\text{eff}}-1)}$ to leading order, and a fit to that stretch returns one number. The correction terms differ, and in principle a fit sensitive enough to resolve them would separate the two: one falls off as a power of y_c/y and the other reflects how the prefactor grows. In practice the terminal stretch is short, the data there are few, and the corrections are the part a fit determines least well. **It cannot separate a damping term from a growing prefactor**, since neither survives in the limit it is fitted to. **The information distinguishing them is in the quiet phase, where one decays and the other does not**, and this is the formal counterpart of the argument above.

A third argument reaches the same place from outside the dynamics altogether, and it bears on the value 2 specifically. Part II, Section 5.2 shows that whatever the regime, the supply a closing gap demands scales as G^{2-m} : for $m < 2$ it falls away, for $m > 2$ it diverges, and $m = 2$ is the only exponent a driver of bounded capacity can sustain to the end. **Any physical system whose driver is finite is therefore pushed toward 2 in its terminal stretch**, not because its dynamics are quadratic but because nothing else is deliverable there.

This makes the measured quadratic behaviour weaker evidence than it appears. The established treatment derives $m = 2$ from the second-order term of a weakly nonlinear expansion, which is a statement about the model; the argument above gives the same value for any bounded driver, which is a statement about energetics and holds whatever the model. **The two are consistent with each other and with a good many other things**, and a measurement returning 2 at the eruption discriminates among none of them. **The exponent at the eruption is therefore not merely derived rather than measured**, as Section 1.1 began by observing; even measured, it would not be the quantity that separates the readings.

2. POSSIBLE CONNECTIONS: ACCELERATING CREEP AND SLOPE FAILURE

The strongest correspondence found so far between the structure of this paper and an established body of physical measurement does not come from any of the domains listed in Part II, Section 1. It comes from geomechanics, and it is worth setting out precisely, because **the equation was written down independently, is in operational use, and its exponent is measured rather than assumed.**

Voight's relation. In the terminal stage of material failure, the accelerating creep of a slope, a rock mass or a loaded specimen is commonly described by

$$\ddot{\Omega} = A \cdot \dot{\Omega}^\alpha$$

where Ω is the observed displacement, the dots denote time derivatives, and A and α are empirical constants (Voight, 1988, 1989; building on Saito and on Fukuzono, 1985). **For $\alpha > 1$ this relation predicts divergence of the velocity at a finite time**, which is read as the transition from accelerating creep to complete failure. Setting $y = \dot{\Omega}/v_{ref}$ and $x = t$ turns it into $dy/dx = A' y^\alpha$, which is **the equation of Part II, Section 2 with the constant term removed and with α in the role of m** . The critical boundary is the same value, 1, and it is crossed for the same reason.

The exponent is measured, and it lies above the boundary. Voight's compilation gives α in the range 1.74–2.01 for metals, 1.90–2.11 for soils, and close to 2 for natural landslides; a more recent compilation spans roughly 1.53–2.15, with the prefactor A varying across three orders of magnitude. This is the first candidate encountered in this work where **the supercritical exponent is an outcome of measurement rather than an assumption of the model. What that measurement does not settle is which of two descriptions applies.** By Part II, Section 4.3, a singularity is completed only during an excursion above the critical value, so an exponent fitted to the terminal approach of a completed event is drawn from above the boundary whatever distribution it came from. A measured $\alpha \approx 2$ is therefore consistent with a fixed exponent of 2, and equally consistent with a fluctuating one whose mean lies below the boundary. **The measurement establishes that the ending was supercritical, which was the only result it could have returned, and leaves the rest open.**

The inverse-velocity method is the $\alpha = 2$ case of Proposition 3. Part II, Section 2 gives the distance still required to reach the singularity from a level y as $T(y) \approx y^{1-m}/[k(m-1)]$. At $m = 2$ this is simply proportional to $1/y$, so the remaining time is a linear function of the inverse velocity. This is exactly the standard forecasting procedure in the field: plot $1/\dot{\Omega}$ against time and extrapolate to zero. Applied retrospectively to the Vaiont slide, Voight noted that the failure date could have been identified more than ten days in advance by this route. **What is a forecasting heuristic there is, in the present framework, a corollary of the blow-up integral.**

A mathematical addition: the constant term restores the earlier stage. Voight's relation describes only the accelerating, tertiary stage; it says nothing about the steady, secondary creep that precedes it, and it cannot, since at $\dot{\Omega} = 0$ it predicts no motion at all. The equation of Part II, Section 2, written for this case as

$$\ddot{\Omega} = k [1 + (\dot{\Omega}/v_{ref})^\alpha]$$

contains both stages in one expression. While $\dot{\Omega} \ll v_{ref}$ the constant term dominates and the creep is steady; once the velocity passes the reference scale the feedback term takes over and Voight's relation is recovered as the asymptotic form. **The exponent is written as a constant here because Voight's relation is, and because the comparison is with that relation.** Nothing in the generalization requires it: the same expression with α varying is the equation of Part II, Section 3, and for the reason given above a measurement of a completed event could not distinguish the two. The crossover velocity v_{ref} then becomes a measurable quantity in its own right, and the blow-up time follows from the closed form of Proposition 1 rather than from a fit restricted to the final phase. Whether this single-expression form fits creep data better than the two-stage description is an open, and directly testable, question.

An observation about the measured values. Integrating the local scaling law of Part II, Section 2 shows that the velocity diverges at t_f for every $\alpha > 1$, but the *displacement* reached at that moment is finite only for $\alpha > 2$, and diverges for $\alpha < 2$. **The measured exponents cluster at almost exactly this dividing value.** No explanation is offered here, and the clustering may well be an artifact of fitting procedures that assume $\alpha = 2$ in the first place. It is recorded because it is the kind of coincidence that is either meaningless or important, and the distinction is decidable from data.

Aborted accelerations. The field also records precursory accelerations that do not complete. The 1986–87 acceleration of the La Clapière landslide is one documented case, and a second occurred within weeks of a failure: at Mud Creek in 2017 the slope moved some three metres between early February and mid-March, slowed again, and only began its runaway acceleration about thirty days before collapse (Handwerker et al., 2025). **The authors note the consequence for forecasting directly**, since their model accounts only for a single monotonic increase in rate, so the earlier episode registers as a false alarm. Such episodes are normally treated as false alarms, or as failures of the forecasting method. **In the framework of Part II, Section 3 they are neither: they are excursions above the critical boundary that ended before the required duration had elapsed.** If that reading is correct, the archives of this field already contain what is hardest to obtain elsewhere, namely a record of both completed and uncompleted excursions in the same system, which is precisely what the waiting-time predictions of Part II, Section 3 require in order to be tested.

A candidate for the counter of Part II, Section 4.2.1 can be named here, which is more than could be said before. The events would be the individual localized slips within the mass: a portion of the body moves, and the load it was carrying is transferred to the material around it, which

has not yet moved. **Load transfer is irreversible in the relevant sense**, since the material that released its share does not take it back, and the portions that received it are now closer to their own limits. The creeping of the slope would then be the visible consequence of many such transfers rather than a quantity in its own right.

This would satisfy the condition in the weak form that section gives it. The slips are countable in principle, their effects are cumulative, and their weights are positive, which is all the condition requires; it does not require that anyone count them. **It also accounts for the anticorrelation of Part II, Section 4.3.2 directly:** a sensor on a portion that has just released reads a decrease at the moment its neighbours read an increase, and the total rises throughout. **What initiates the first slip need not be a fluctuation in any exponent:** a local threshold reached by pore pressure after rainfall would do, which is the metastable initiation of Section 1 rather than the mechanism of Part II, Section 3, and the cascade proceeds from there on its own.

None of this is established, and it is offered as the form an answer would take rather than as the answer. Load transfer between portions of a failing mass is standard in this field and is not introduced here; what is conjectural is that the transfers constitute the counter the construction requires, and that their rate depends on the accumulated total in the way the equation needs. **Both are questions for the field, and both are of the kind that measurement could settle**, since acoustic emission already records events of this type even though it cannot record all of them.

A slope that stops need not have failed to accelerate. The aborted accelerations recorded in this field admit a second reading: not an excursion that began and was not sustained, but a lower level of the structure that completed, reached its own bound, and terminated there. **The two leave different states behind.** After an interrupted excursion the mass is where it was; after a completed level the transfers it made have already occurred, and whatever resumes begins from the higher accumulated total. **Neither reading is established, and the records available do not obviously separate them**, since both show acceleration followed by quiescence.

On the second reading a long dormancy is not evidence of stability. The structure continues below the level that completed, at rates too slow to register, and a further initiation of the kind described above would resume the cascade from where the accumulated total now stands. **A slope quiet for a century would then be closer to failure than one that had never moved**, which is the opposite of what quiescence ordinarily suggests. **This is a consequence of the construction and not a claim about any slope**, and it is recorded because it is testable in principle: it predicts that the interval before the next acceleration shortens as the transfers accumulate, which is a statement about the history of a site rather than about its present state. **The test is harder than it sounds, for the reason set out in Part II, Section 5:** the contraction is weak and the events few in exactly the phase a long record covers, so an early record of this kind will fit a memoryless process whether or not the structure is present. **Published recurrence statistics bear on this.** Repeat movement at the same site is well documented, and one analysis of large submarine slides across several basins found intervals close to a Poisson distribution, with the probability of the next event independent of the time since the last. **That result is the one the preceding paragraph would have to contend with**, with the qualification that it aggregates events across sites and over intervals of thousands of years, which is the regime in which the contraction would be least visible.

What is not claimed. The correspondence above is between equations, not mechanisms, and the membership conditions of Part II, Section 4 are not satisfied at present. Two gaps are specific enough to state. **The exponent is not among those gaps**, for the reason given above: no measurement of a completed event could have returned a subcritical value, so none is expected and its absence excludes nothing. The gaps lie elsewhere. First, **the non-decreasing state variable has not been identified.** It is not the velocity: that can and does fall again, as the aborted accelerations themselves demonstrate, but nothing in the construction requires it to rise monotonically. **An observed quantity is expected to fall**, both in the nested simulation of Part II, Section 4.3, where every level drops back at each completion, and for the sampling reason of Part II, Section 4.3.2, where a measurement of part of an extensive quantity may move against the whole. The condition constrains the irreversible quantity instead, and a candidate for it was named above: the cumulative load transferred by successive localized slips, which is a counter in the sense of Part II, Section 4.2.1 rather than a quantity anyone reads off an instrument. **What is missing is not a candidate but a demonstration**, that these transfers are in fact cumulative and irreversible in the relevant regime, and that their rate depends on the accumulated total in the way the equation requires. **This has moved from an unidentified variable to an unverified one**, which is a smaller gap and a different kind of question. Second, **it is not established which physical quantity fluctuates the exponent, though less turns on this than it appears.** What must be identified is the coupling between levels, since by Proposition 5 a transfer reaching only the rate of a level cannot produce a singularity there. **The initiating disturbance is a separate matter and need not reach any exponent at all:** a local threshold crossed by pore pressure would start the first slip without altering a feedback exponent anywhere, after which the cascade supplies its own fluctuation. **The same holds in the idealized circuit of Part II, Section 6.3**, where the base fluctuation is thermal, acts on a state variable rather than on an exponent, and could not be identified as an exponent-driver because it is not one. **What remains to be established here is therefore the coupling and not the trigger**, and the question of which quantity fluctuates the exponent applies to the former. The same filter applies here as everywhere in this paper: rainfall and pore-water pressure act principally on the prefactor A , and prefactor fluctuations cannot produce a singularity; it is damage accumulation and the evolving crack network that would have to move α itself. Whether they do so, and whether they carry it across the boundary intermittently, is unresolved.

Why the section is included. Voigt's relation is phenomenological: it is fitted, widely used, and not derived from first principles. A framework that specifies where such an exponent comes from, why it sits above 1, and what governs its variation would be addressing an open question in that field rather than importing a solved one. **That is the reason this direction is recorded here, and it is recorded as a correspondence between equations and nothing more.**

3. POSSIBLE CONNECTIONS: THE LIGHTNING STEPPED LEADER

This candidate meets more of the conditions of Part II, Section 4.2 than any other considered here, and it is set out for that reason. The correspondence was identified from the side of electronics, where the governing quantities are the same ones that appear in discharge devices, high-voltage supplies and dielectric breakdown, and the account below rests on that. **What it does not rest on is field measurement of lightning**, which this author has not done and cannot assess, and which is where the question would be settled. **It is offered as a candidate for people who make those measurements**, and until they have examined it, it is not a member.

The arrangement. A negative cloud-to-ground leader descends in discrete steps rather than continuously, each step extending an ionized channel toward the ground. **The channel, once formed, is conducting**, so the cloud potential is carried down to the tip of the extension and the field that drives the next step acts across the remaining distance rather than across the original one. **The gap is therefore the distance still to be bridged**, and it closes in steps, each of which shortens it and none of which lengthens it.

The membership conditions, taken in order. The non-decreasing quantity of Section 4.2.1 is the extended channel, and it is irreversible for a physical reason rather than an assumed one: a segment that has become conducting stays conducting for the duration of the event. **The counter is the step count**, which is observed directly rather than inferred. The transfer is positive, each step is small compared with the distance remaining for most of the descent, and a step does not return the system to its origin, which are the three conditions of Section 8.5.1.

The feedback is structural and does not have to be posited. With the channel conducting, the field across the remaining gap is of order V/a , and the rate at which a step forms depends on that field; the channel thus accelerates its own extension by shortening the distance across which its own potential acts. **If the step rate varies as a power of the field**, so that the closing speed goes as E^α , then

$$\frac{da}{dt} = -c a^{-\alpha} \rightarrow m = 2 + \alpha$$

by the relation of Part II, Section 6.2. **The exponent would then be a measured quantity rather than a derived one**, obtained from the step record in the same way the creep exponent of Section 2 is obtained from displacement data.

The field at the tip is the step that makes this concrete, and it is elementary electrostatics rather than anything belonging to this paper. A conducting channel of length $L(t)$ descending toward ground across a total separation D leaves a gap $a(t) = D - L(t)$, and the field at its tip is

$$E_{tip}(a) \approx \frac{U_0}{a \ln(4D/r_{tip})} \propto a^{-1}$$

so the quantity driving the next step is the reciprocal of the gap, which is the mapping of Part II, Section 8.5 appearing as a physical fact rather than as a choice of variable. With a propagation speed depending on that field as a power, $v_L = k E_{tip}^\alpha$, substitution gives the closing law above directly. **The logarithm is not constant**, since r_{tip} and the effective D both change during a descent, so a measured exponent should be expected to drift slowly for that reason alone, which is one more entry in the list of Section 8.

The same motion read as a speed rather than as a gap gives a different exponent, and both are correct. Direct differentiation of the speed $v_L = c a^{-\alpha}$ via the chain rule yields:

$$\frac{dv_L}{dt} = \frac{dv_L}{da} \frac{da}{dt} = (-\alpha c a^{-\alpha-1}) \cdot (-c a^{-\alpha}) = \alpha c^2 a^{-(2\alpha+1)}$$

Substituting $a = (c / v_L)^{1/\alpha}$ eliminates the gap, recovering the autonomous feedback equation directly in speed space:

$$\frac{dv_L}{dt} = \alpha c^{-1/\alpha} \cdot v_L^{2 + 1/\alpha} \rightarrow m = 2 + \frac{1}{\alpha}$$

The identical result follows from the terminal scaling law of Part II, Section 6.2. Integrating the closing law gives $a(t) = [c(\alpha + 1)(t^* - t)]^{1/(\alpha+1)}$, fixing the gap exponent to $p = 1/(\alpha + 1)$:

$$y = 1/a: \quad q = 1 \rightarrow s = \frac{1}{\alpha+1} \rightarrow m = 1 + \frac{1}{s} = 2 + \alpha$$

$$y = v_L: \quad q = \alpha \rightarrow s = \frac{\alpha}{\alpha+1} \rightarrow m = 1 + \frac{1}{s} = 2 + \frac{1}{\alpha}$$

$$y = 1/a: \quad q = 1 \rightarrow m = 2 + \alpha \quad y = v_L: \quad q = \alpha \rightarrow m = 2 + 1/\alpha$$

This is the observation map of Part II, Section 4.3 in a physical system rather than a constructed one. Two instruments trained on the same descent, one recording the distance remaining and the other the speed of the tip, return exponents that differ unless $\alpha = 1$, and neither is wrong. **Any exponent quoted for this system has to say which quantity was measured**, and the two coincide only in the case where the speed is proportional to the field.

Both forms exceed 2 for every positive α , and Part II, Section 5.2 then applies with force. There the supply demanded by a closing gap scales as G^{2-m} , which diverges above $m = 2$; here that supply is the current reaching the tip, and no cloud delivers it without limit. **The description must therefore fail before the gap closes**, and the point at which it fails is set by how fast charge can be moved to the tip rather than by the field there. This is not a reservation about the correspondence but a prediction following from it.

The discrete stepping gives the same relation from the other direction. If a step requires a critical charge Q_{crit} to accumulate ahead of the tip and the corona current supplying it grows as the gap narrows, $I(a) \propto a^{-\beta}$, then the interval between steps and the step frequency are

$$\Delta t_{step} = \frac{Q_{crit}}{I(a)} \rightarrow f_{step}(a) \propto a^{-\beta}$$

which is the event-rate law of Part II, Section 8.5 with Q_{crit} in the role of the transfer quantum. If each step advances the channel by a mean step length Δl , the macroscopic propagation velocity is $v_L = \Delta l \cdot f_{step}(a) \propto a^{-\beta}$, directly establishing the equivalence between the continuous field exponent and the discrete charging exponent ($\alpha \equiv \beta$). The number of steps remaining is then the accumulated charge still required divided by that quantum, exactly the count $N = aC/P$ of Section 8.5.1, and the last step overshoots by an amount distributed over one quantum, as that section derives. **Two routes, the continuous and the discrete, give the same divergence**, and the discrete one supplies the grain below which no rate is defined.

The reported phenomenology is consistent with this and does not establish it. Leader speeds range over roughly an order of magnitude, from 10^5 to a few times 10^6 m/s, with a hundred or more steps during a descent lasting tens of milliseconds, and models of the stepping process predict that the speed increases as the tip approaches the ground. **An acceleration toward the end is what the construction requires**, but it is also what any field-driven description predicts, and the question is whether the rate depends on the remaining gap in the specific way written above.

The truncation is known and is of the kind described in Part II, Section 4.1. The divergence is cut off by the speed of light, which is absolute and does not scale with the gap, and before that by the return stroke, which begins once the descending and ascending channels meet. **This is one of the few candidates in which the terminal limit is known in advance rather than inferred**, and where it sits far enough above the measurement resolution to leave a usable range.

Three difficulties remain, and the first is no longer the one it appeared to be. The reduction above is not new: field-dependent propagation laws of the form $v = k E^\alpha$ are standard in leader-progression modelling, and the power-law dependence of growth probability on local field is the defining rule of the dielectric breakdown model of Niemeyer, Pietronero and Wiesmann (1984). **What is not standard is reading these as an instance of the construction of Part II**, and what remains to be established is the value of α for a descending leader, measured rather than assumed. **The difficulty is therefore quantitative rather than structural**, which is a better position than this paper reaches with any other candidate.

The second concerns which quantity carries the anticorrelation of Part II, Section 4.3.2. It is not the channel length: a single record of one quantity has nothing to be anticorrelated with. **The conserved quantity here is charge**, and the comparison would be between what the main channel has drawn and what the branches have taken, or between the remaining cloud charge and the total channel extent. **Which of these can be measured, and how, is a matter of field practice**, and the condition is untestable rather than unmet where no such pair is available. **A separate point concerns which events are recorded at all:** reconstructing a descent backward from the strike selects the path that completed, which is the conditioning of Part II, Section 4.3, so an exponent fitted to it is drawn from events that finished and says nothing about those that did not. Branches form and most of them die, and the charge that went into them did not extend the main channel; by Part II, Section 4.3.2 the branches and the channel are anticorrelated, and a record of the channel alone is a record of one part of an extensive quantity. **Reconstructing the descent backward from the strike selects the path that completed**, which is the conditioning described in Section 4.3: an exponent fitted to it is drawn from events that finished, and says nothing about those that did not.

The third is not a difficulty but a truncation, and it should be stated as one. The treatment takes the driving potential as fixed, which no source is; charge redistributes at a finite rate, branches draw it off unevenly, and the current reaching the tip is bounded. **Since both forms of the exponent exceed 2, Part II, Section 5.2 requires the supply to diverge**, so the description must fail before the gap closes, and where it fails is set

by how fast charge reaches the tip. **This is the same situation as every other realization in this paper**, and it is a prediction rather than an objection: a laboratory discharge driven by a fixed supply should show the same behaviour over the leader phase and depart from it at a point the supply determines. **What follows the departure is the arc, which is a different process and not described here.** The treatment above takes the cloud potential as fixed, which is an idealization: charge redistributes within the cloud at a finite rate, branches discharge it unevenly, and the supply to the main channel fluctuates accordingly. **By Part II, Section 5.2 this is the constraint that matters**, since an exponent above 2 requires a supply that diverges, and the step rate must eventually be limited by how fast charge reaches the tip rather than by the field there.

What would settle it. The step intervals and step lengths of a single descending channel, plotted against the distance remaining, with the slope of the logarithm of the step rate against the logarithm of that distance giving α and therefore m . **High-speed optical records of this kind already exist**, and what the construction adds is a specific quantity to extract from them and a prediction about its value, namely that it exceeds 2 and lies in the range implied by whatever field dependence the field itself accepts. **A measurement showing no dependence on the remaining distance would exclude the candidate**, which is the form in which it should be put. **One caveat about method:** a rocket-and-wire triggered discharge begins with part of the channel already in place, so the gap it closes is not the one described here, and records obtained that way are unlikely to bear on the question.

A negative result should be read with the limitations of this paper's own machinery in view, and not only as an answer about lightning. The simulations do not represent the exponent drift of Section 8.5.3 or the overshoot of Section 8.5, treating the exponent as a two-valued process and the transfer as instantaneous; **they are a guide to the behaviour and not a model of a physical descent.** What the accounting of Sections 8.5.1 to 8.5.3 adds is precisely what the simulations omit, and a candidate should be assessed against that accounting rather than against the figures.

The overshoot in particular may not be a correction here. In the idealized construction it is a small displacement of where the gap closes; in a branching discharge it could initiate a chain, since a step that overshoots carries the channel into a region where the field is higher than the step was formed for. **A real descent may therefore depart from the idealized model in the direction of being faster and less regular**, and a measurement disagreeing with the idealized prediction is as likely to indicate a missing term as a missing correspondence.

Several such terms are evident even from outside the field. The capacitance of the ionized channel, the dependence of its conductivity on humidity at formation, the charge drawn off by branches, and any resonance between branch activity and the redistribution of charge within the cloud would all enter the transfer P_i of Section 8.5.1, which the treatment above takes as constant. **None of these is beyond measurement and none is accounted for here. A result excluding this candidate would therefore be a reason to revisit the membership conditions and the accounting as much as to abandon the correspondence**, and which of the two it called for would depend on how the exclusion came about.

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